






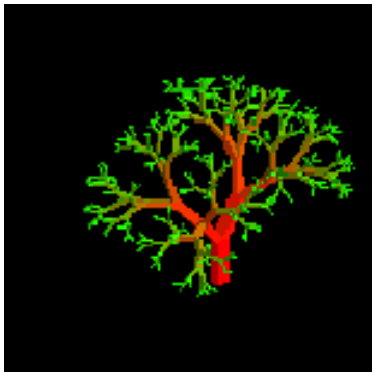







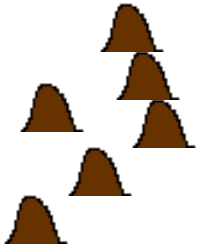





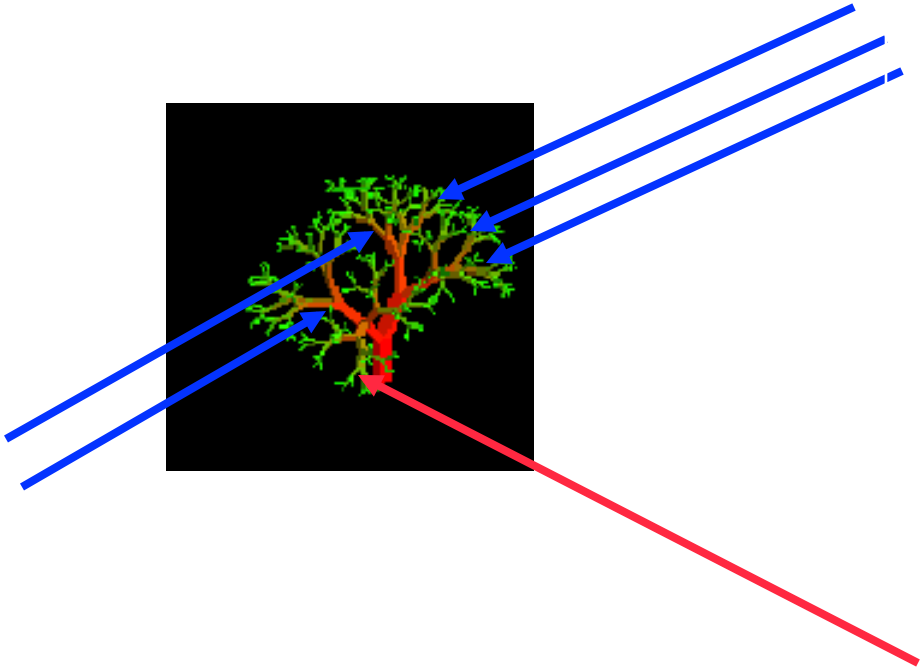

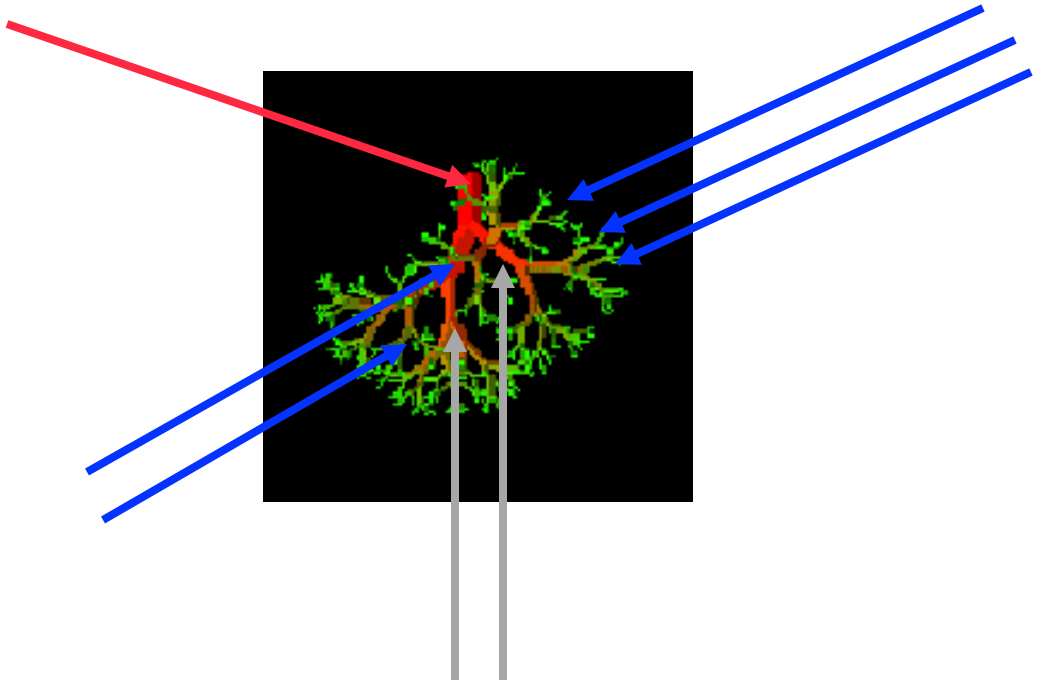

Chapter 5

Trees

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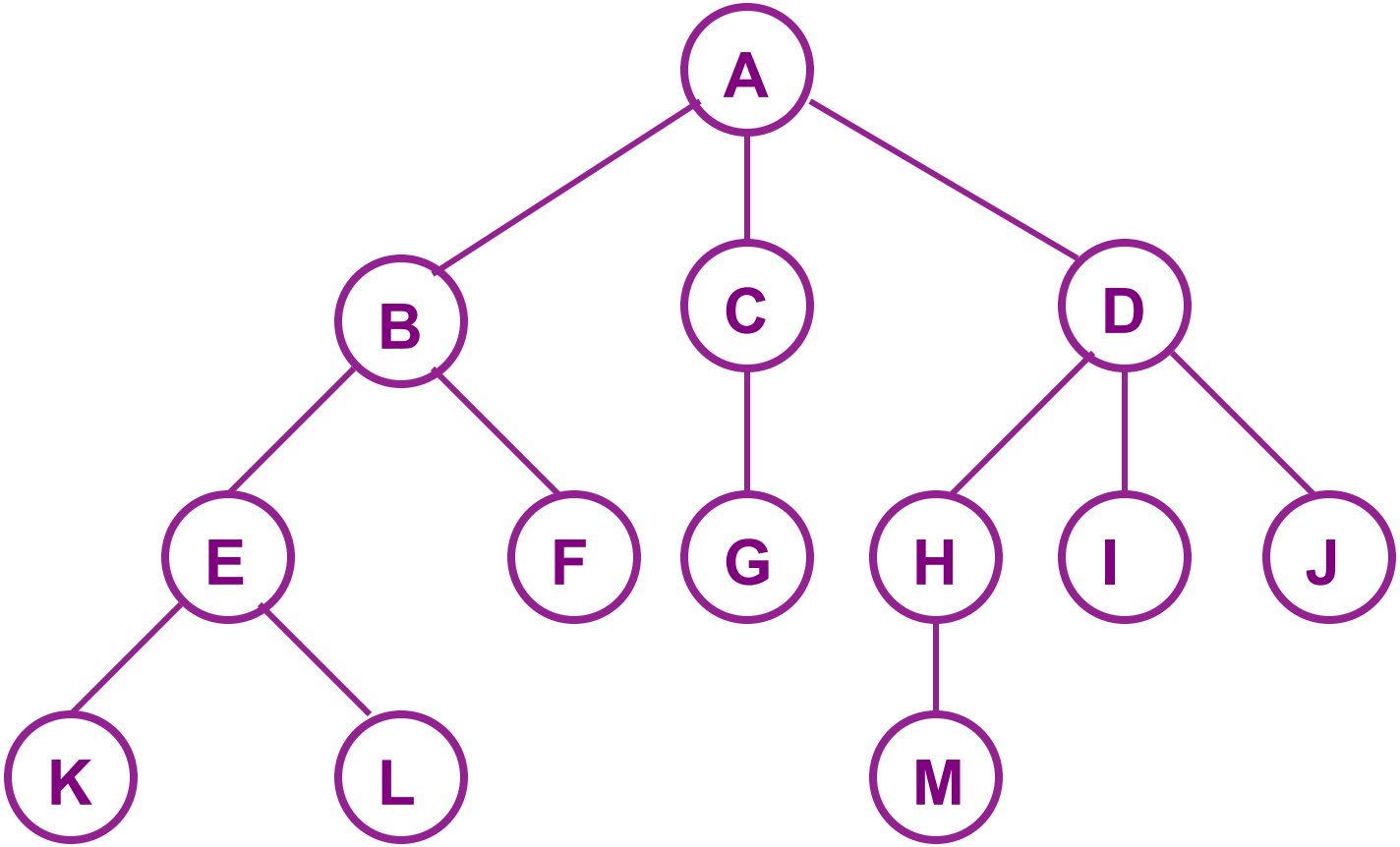
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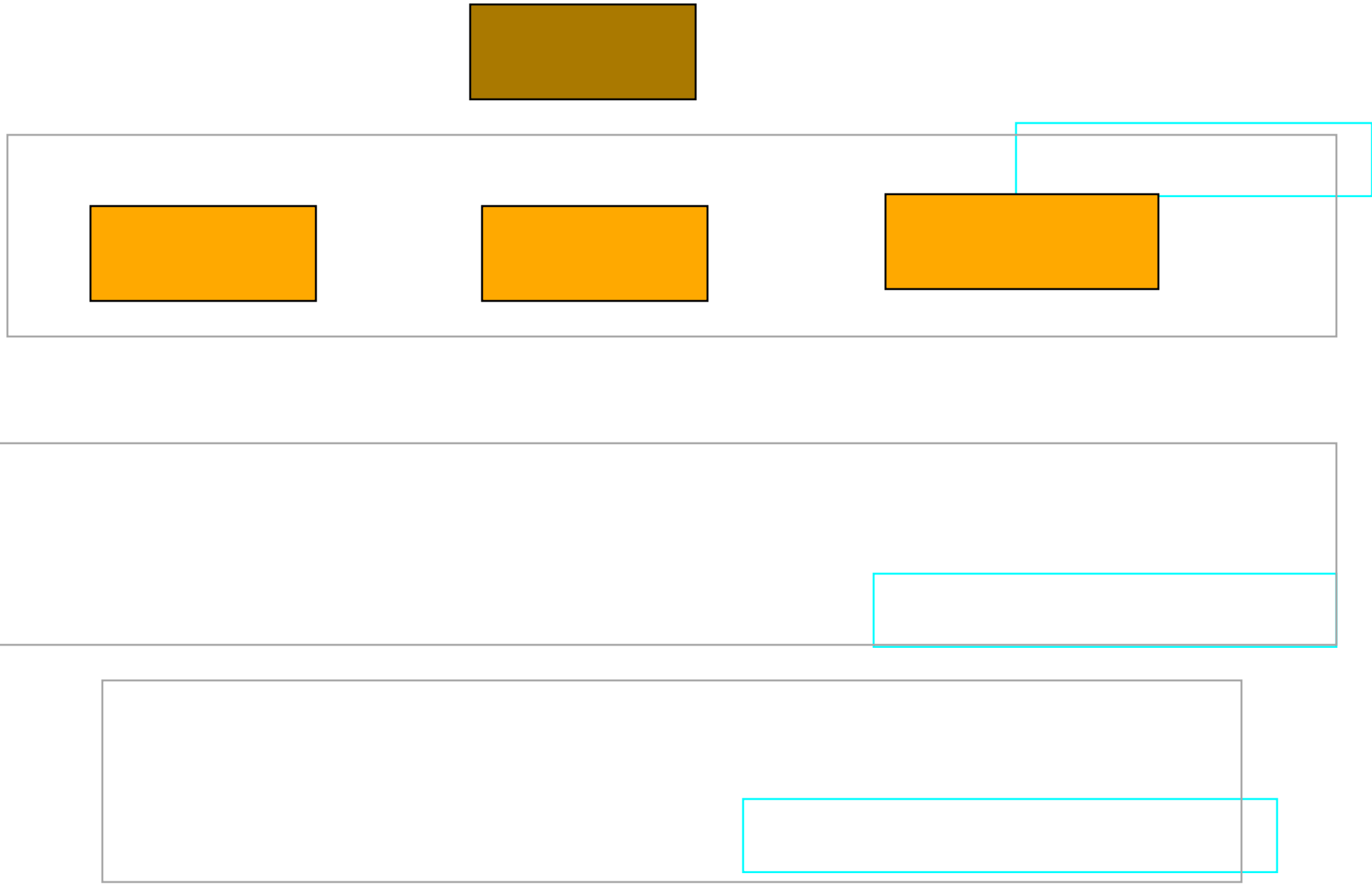

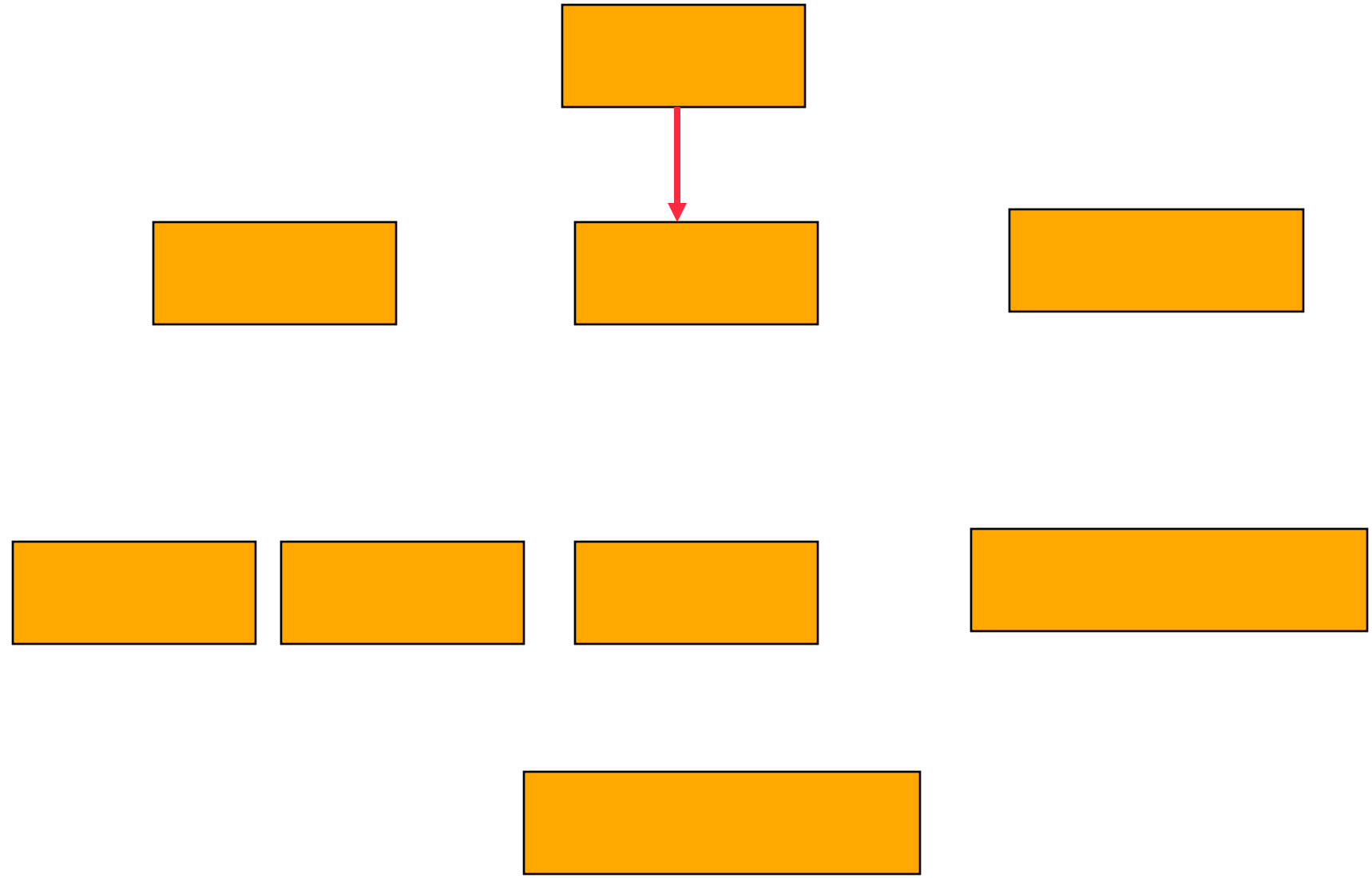

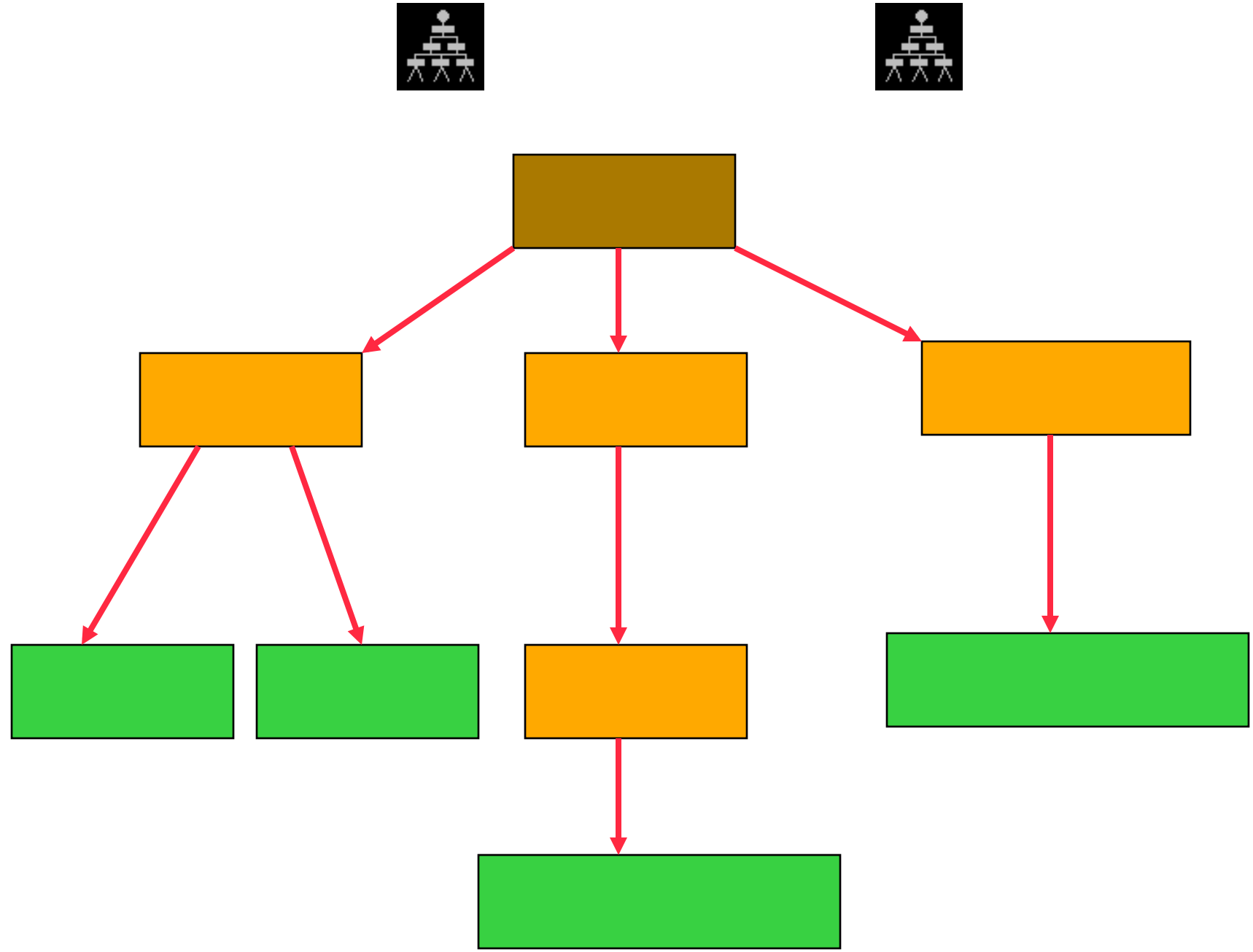

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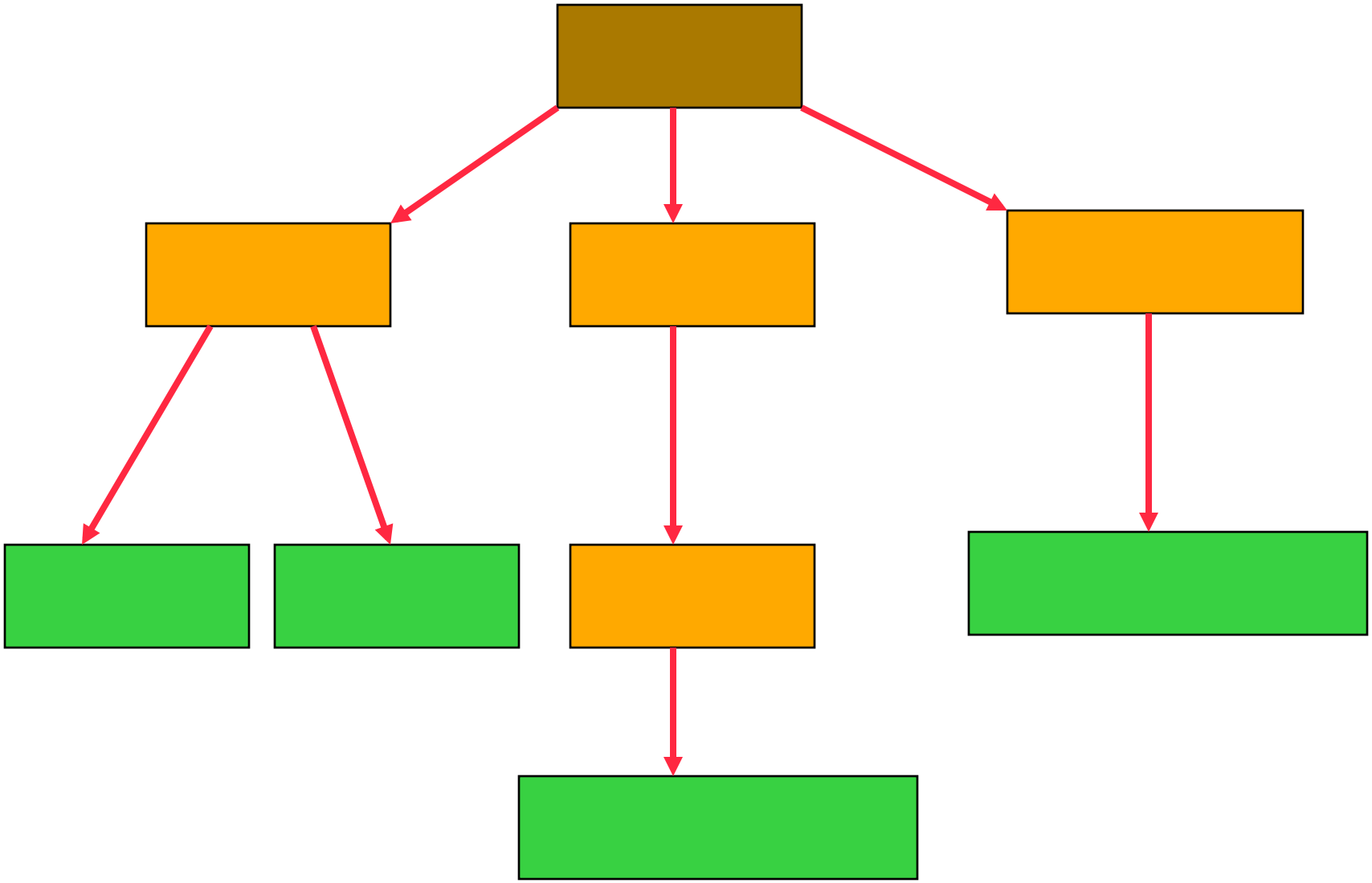

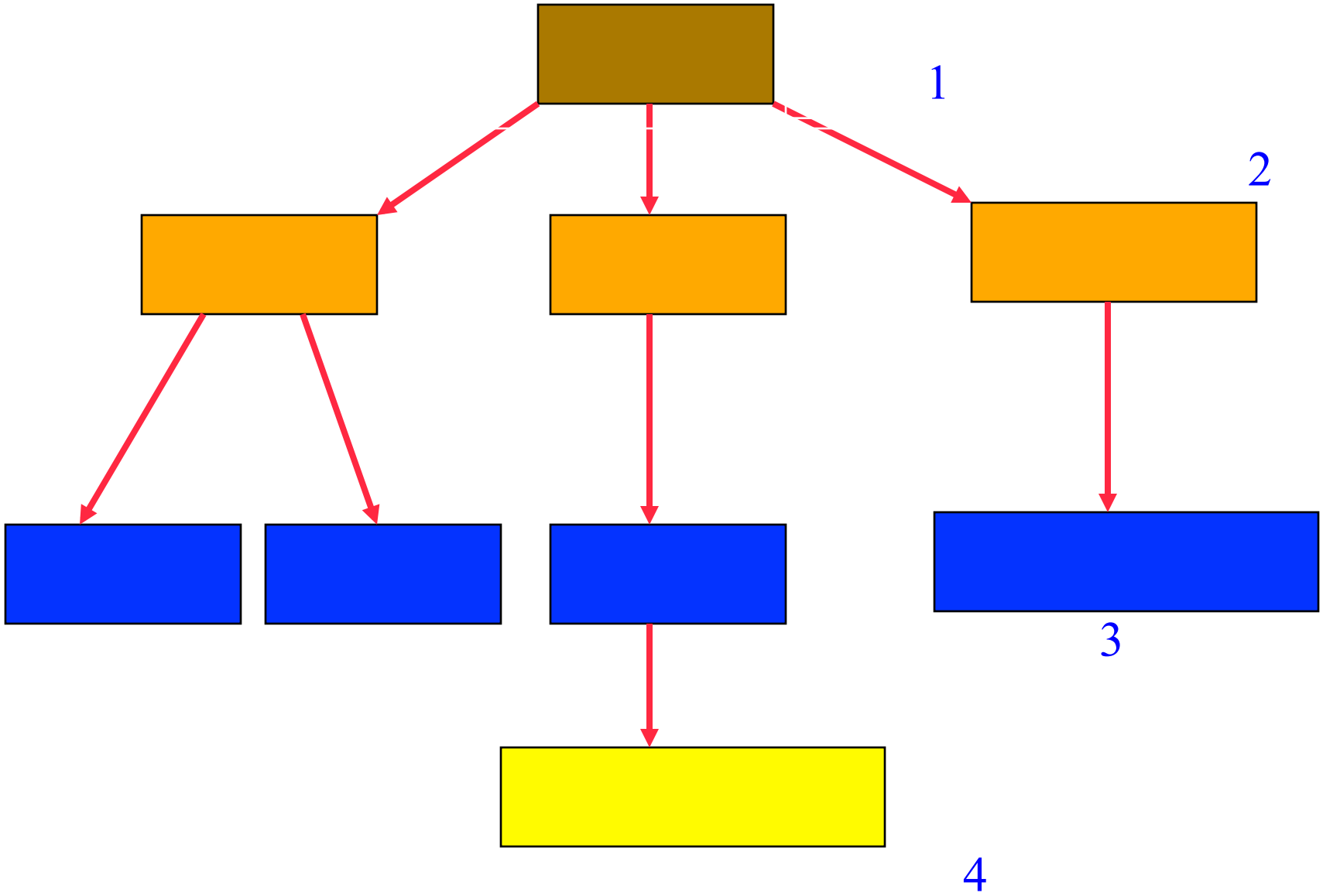
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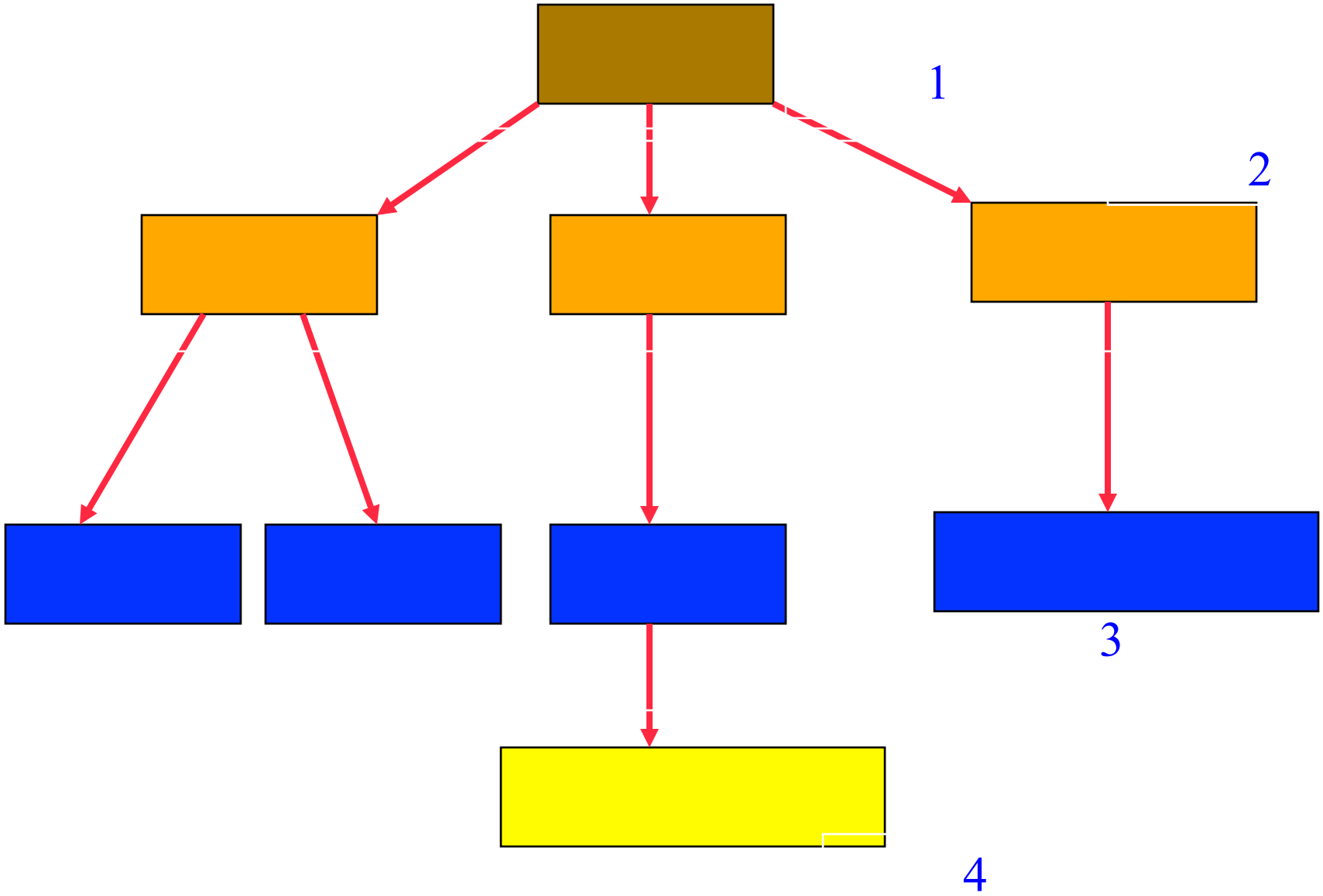
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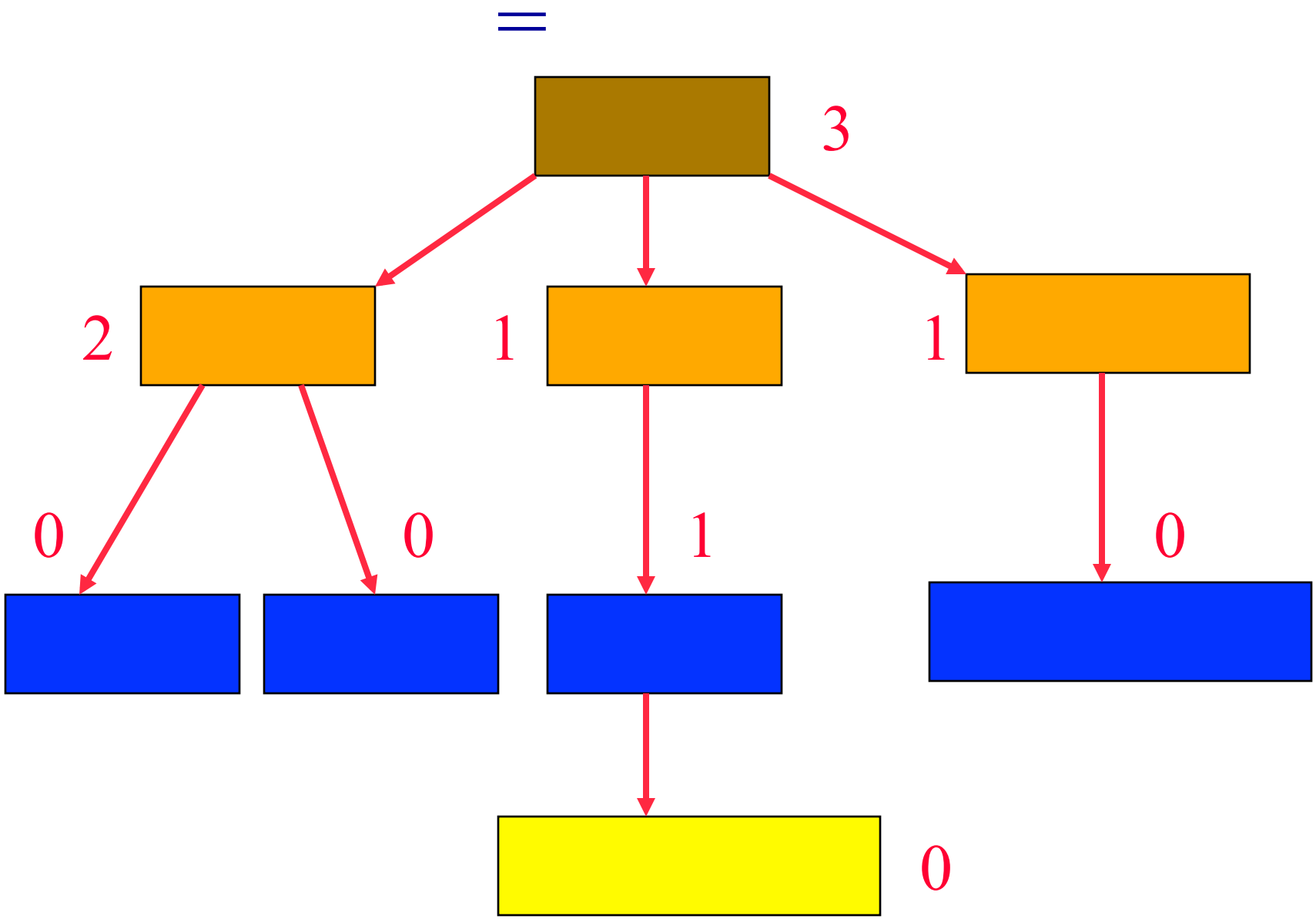
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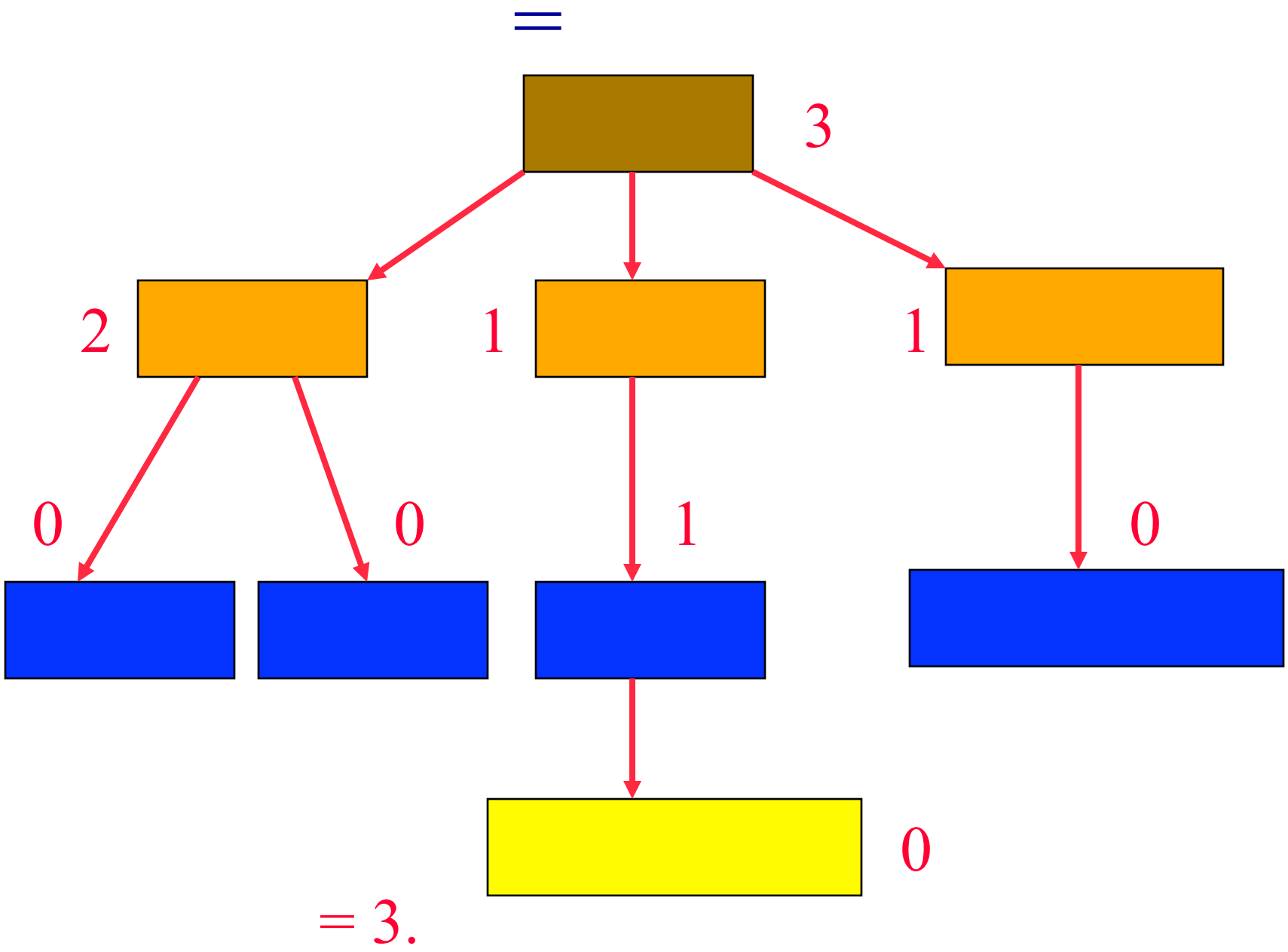
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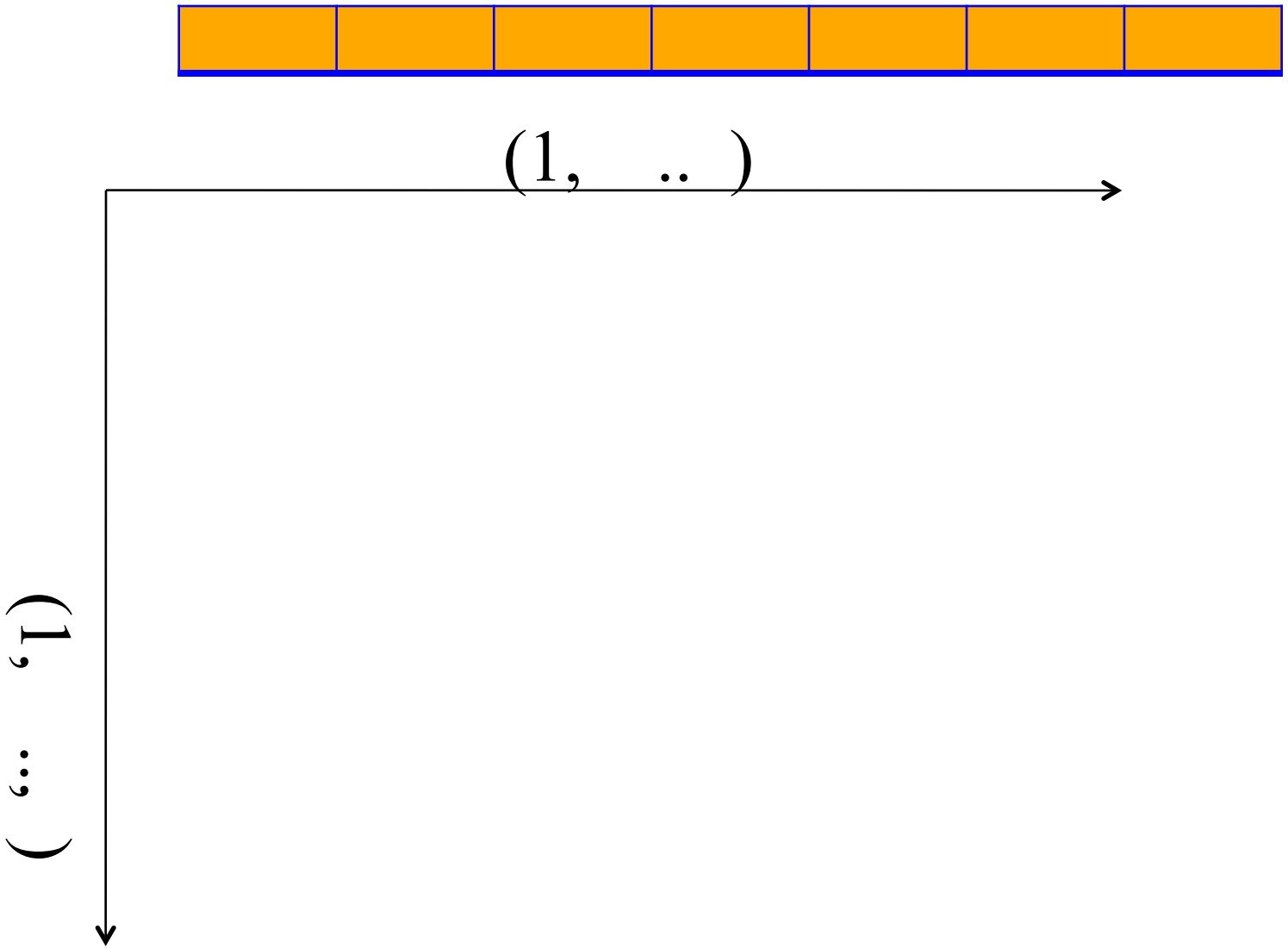
5.1.2 Representation of Trees



5.1.2 Representation of Trees

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Data	Child1	Child2		Child k

Possible node structure for a tree c

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ADT 5.1

```
template <class T>
class BinaryTree
{ // A finite set of nodes either empty or consisting of
  // a root node, left BinaryTree and right BinaryTree.
public:
    BinaryTree ();
    // creates an empty binary tree

    bool IsEmpty ();
    // return true iff the binary tree is empty

    BinaryTree(BinaryTree<T>& bt1, T& item,


BinaryTree<T>& bt2);
    // creates a binary tree whose left subtree is bt1,
    // right subtree is bt2, and root node contain item.
```

```
    BinaryTree  LeftSubtree();
    // return the left subtree of *this

    T RootData();
    // return the data in the root of *this

    BinaryTree  RightSutree();
    // return the right subtree of *this
};
```

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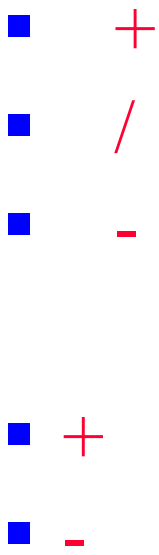

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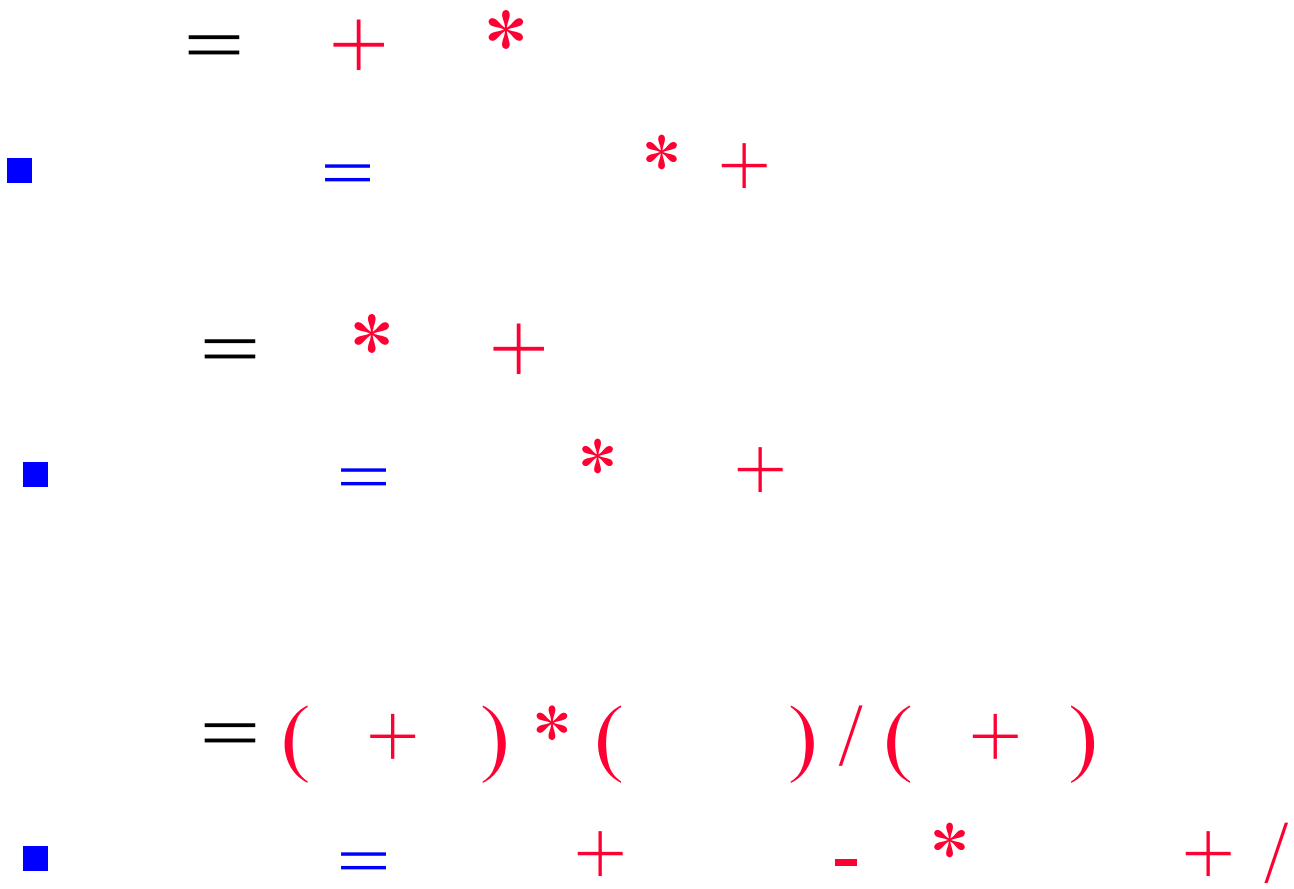

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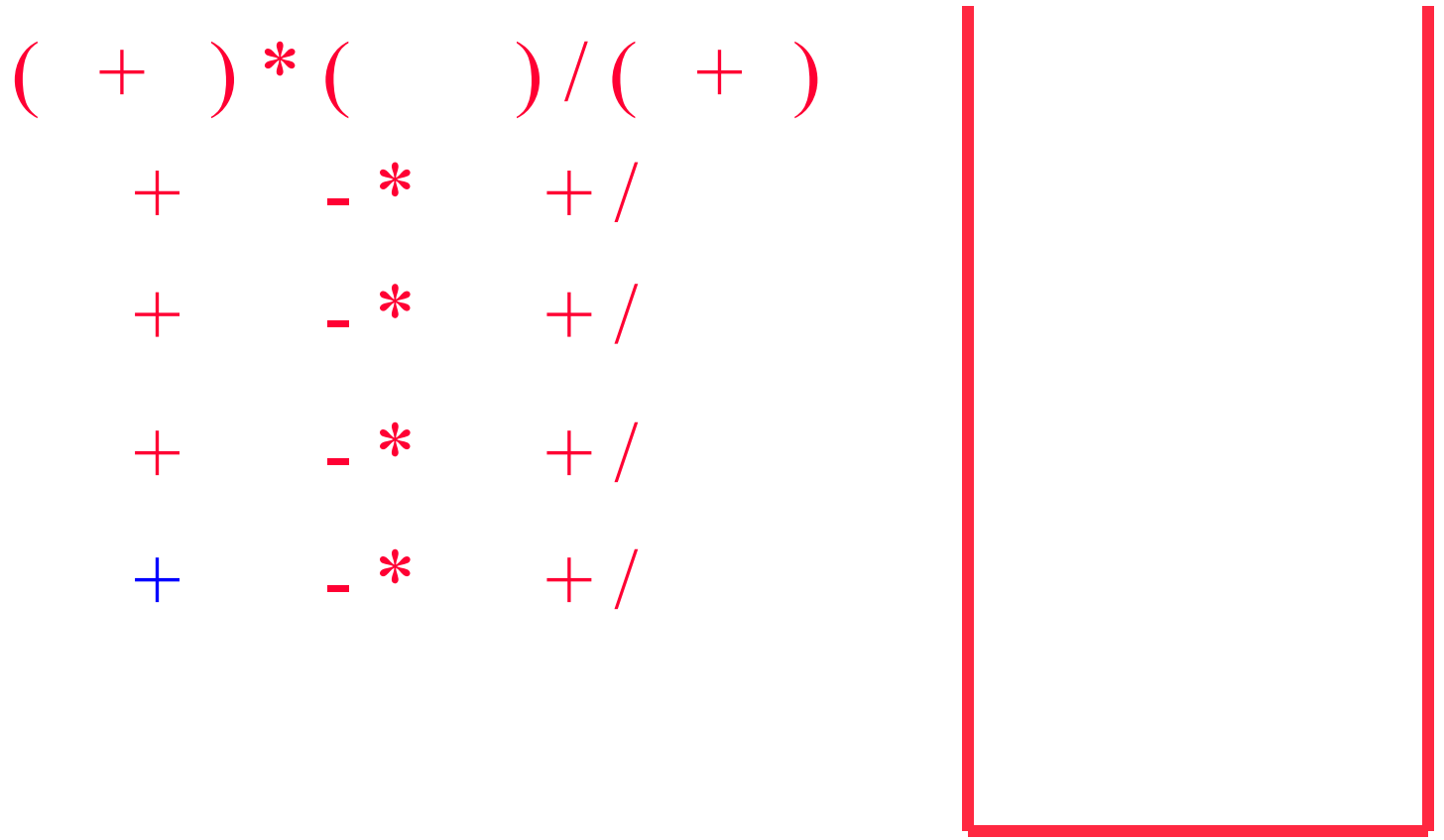

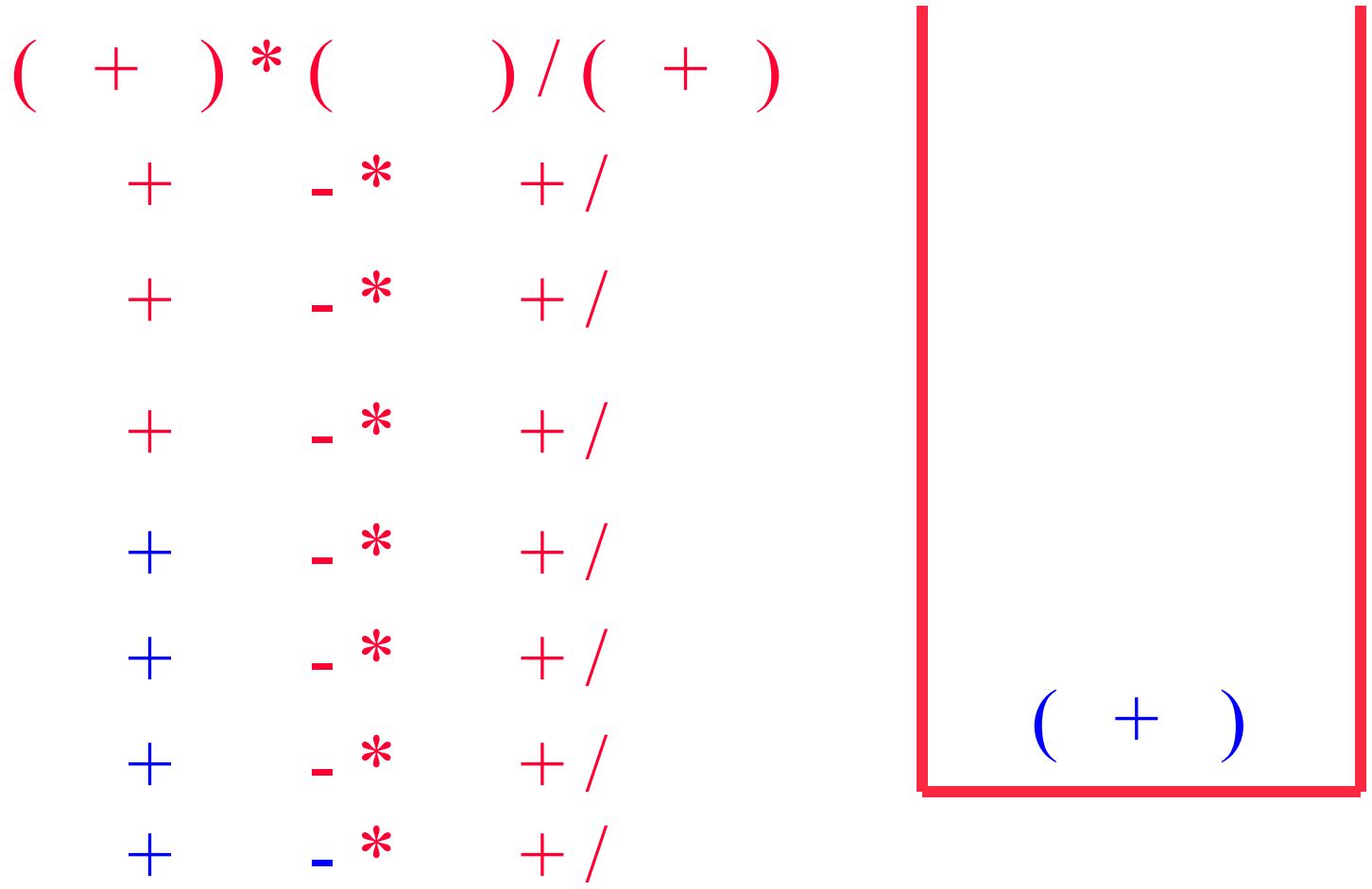

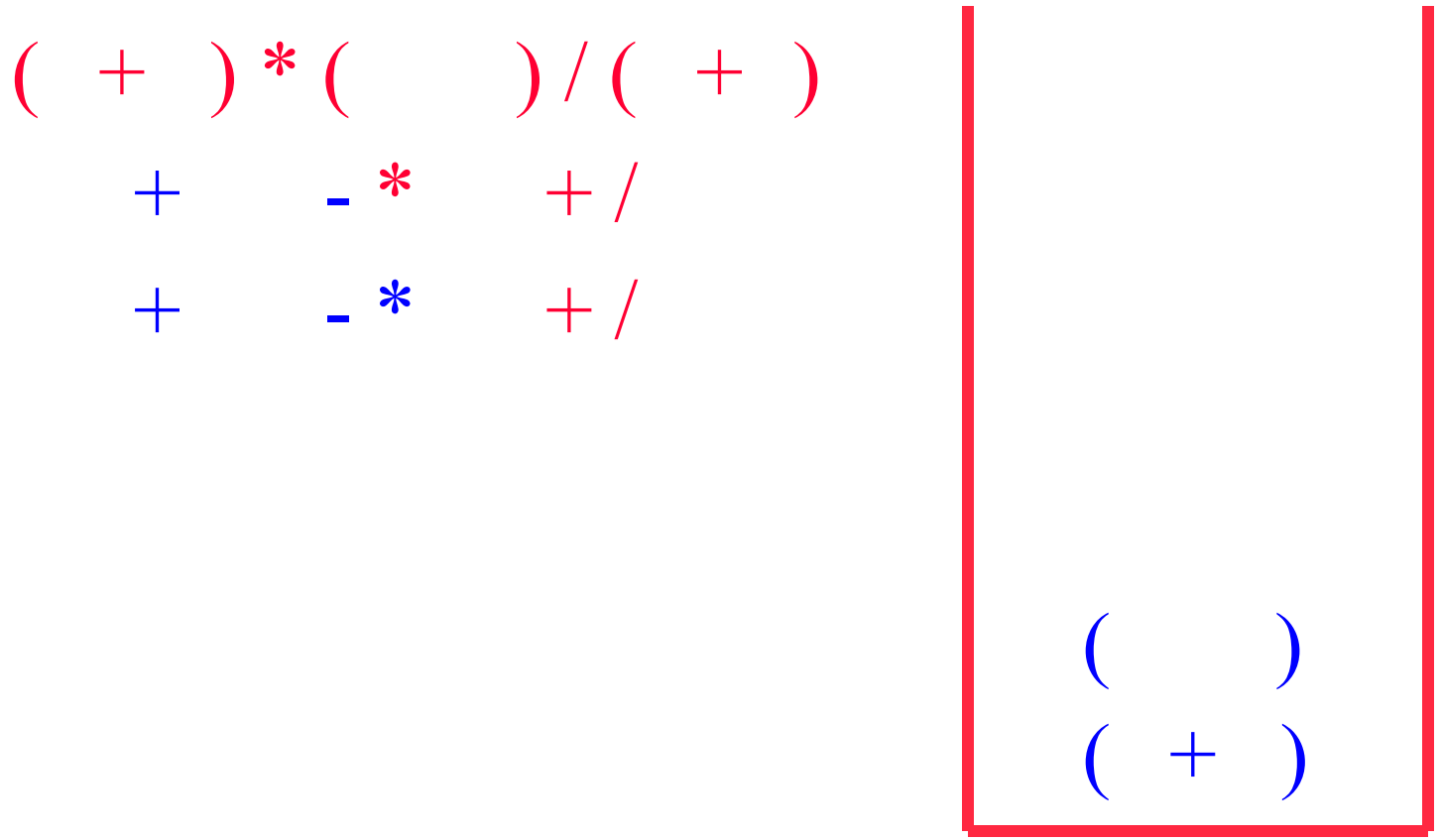

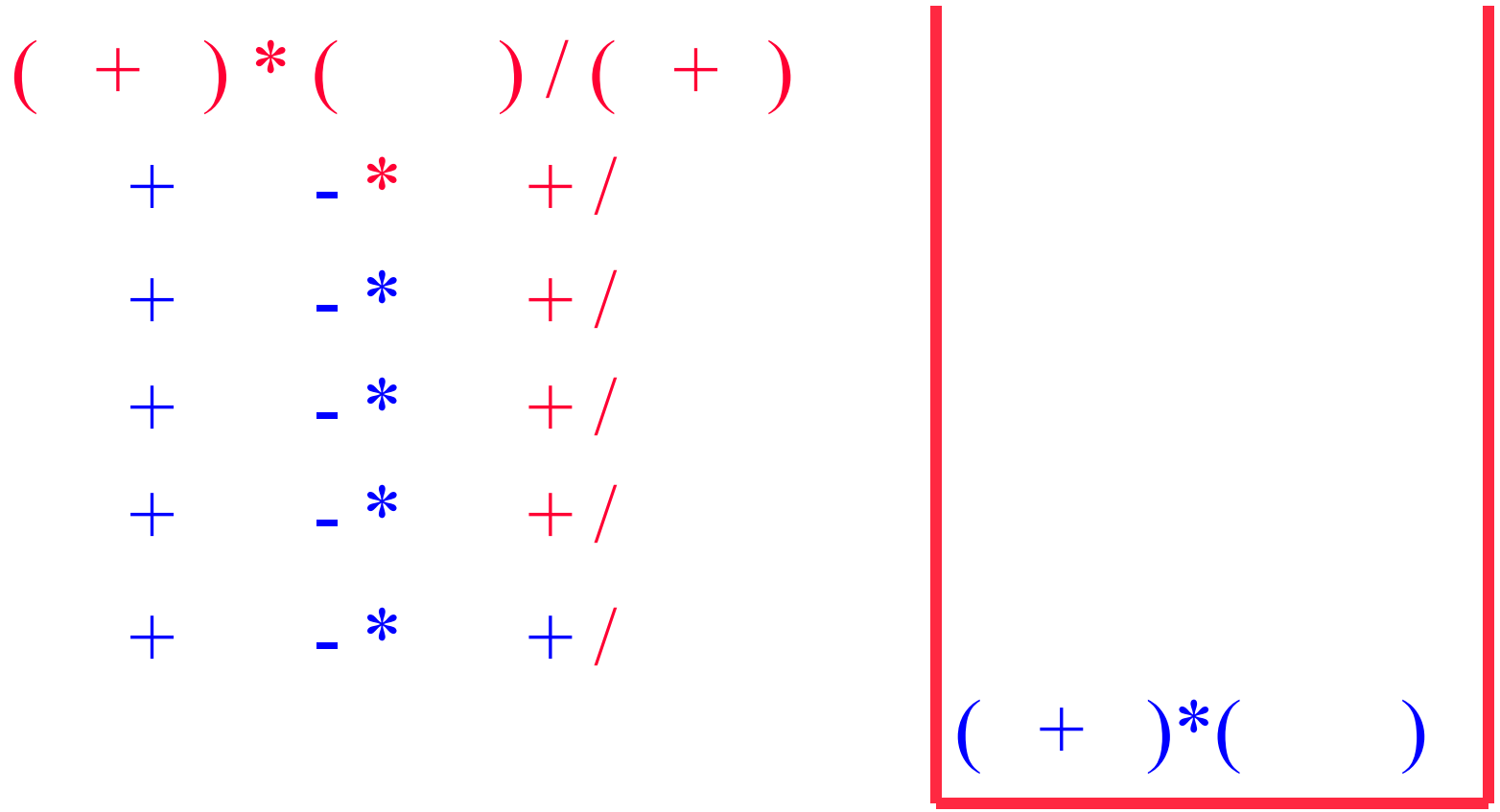

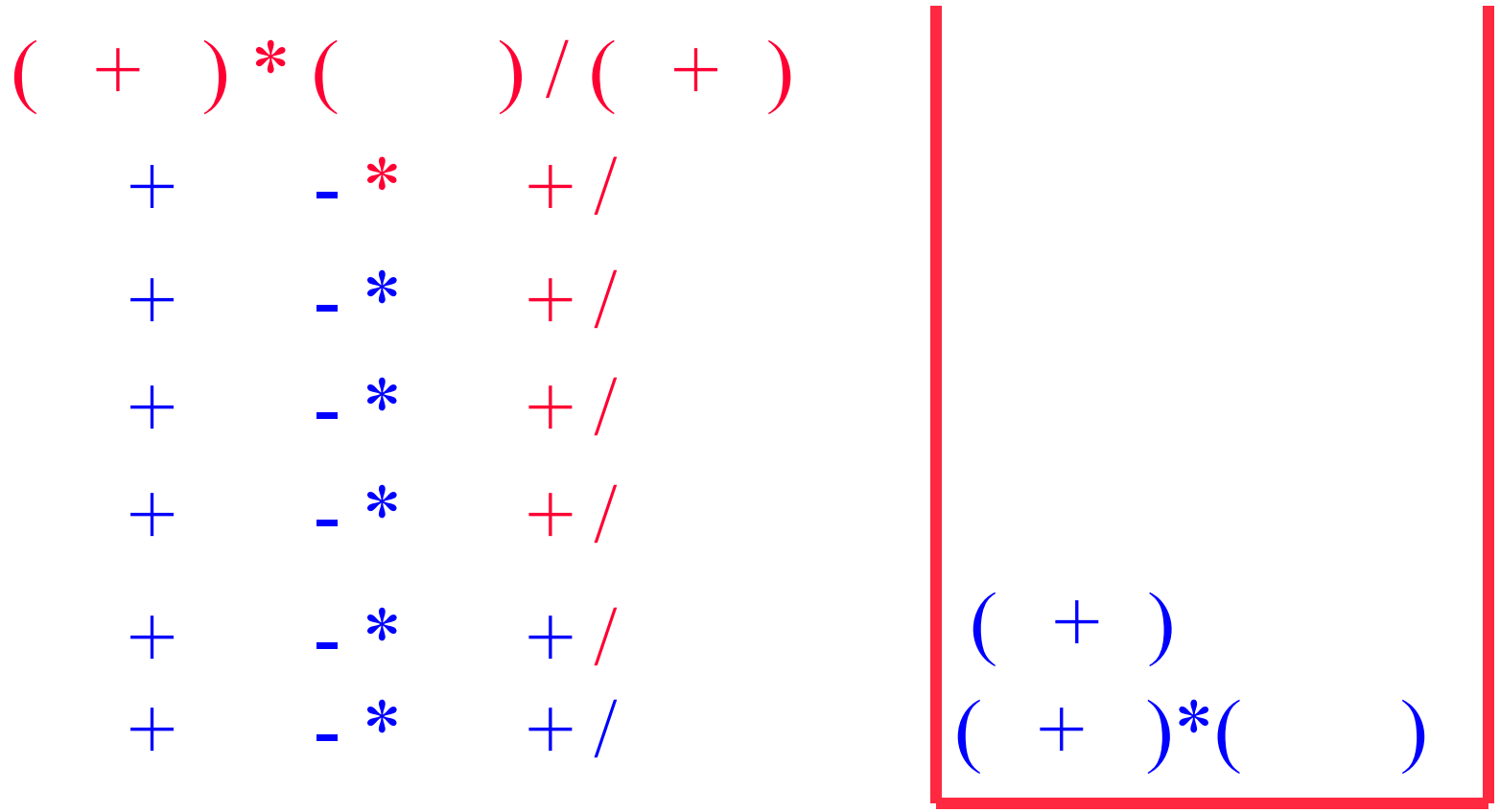

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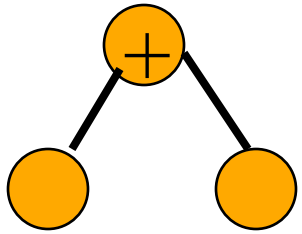

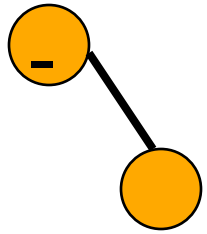

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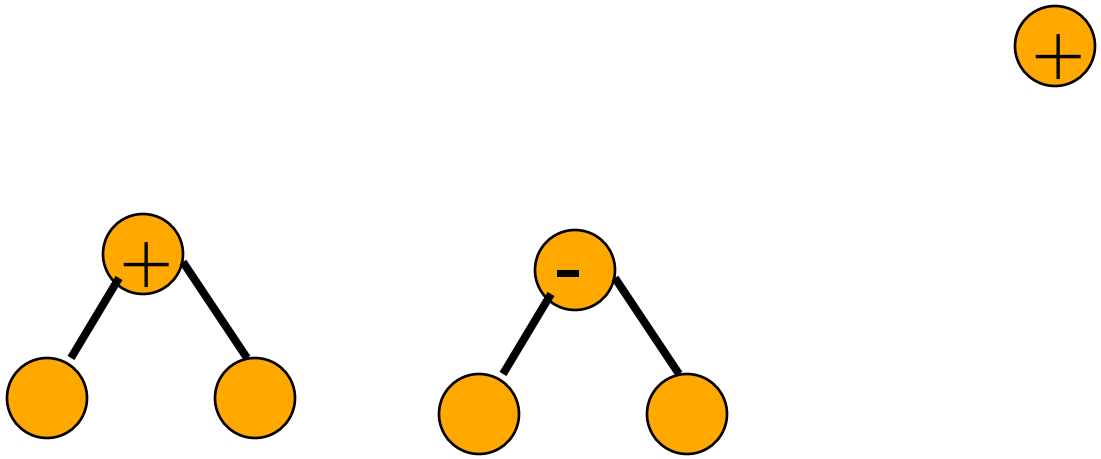

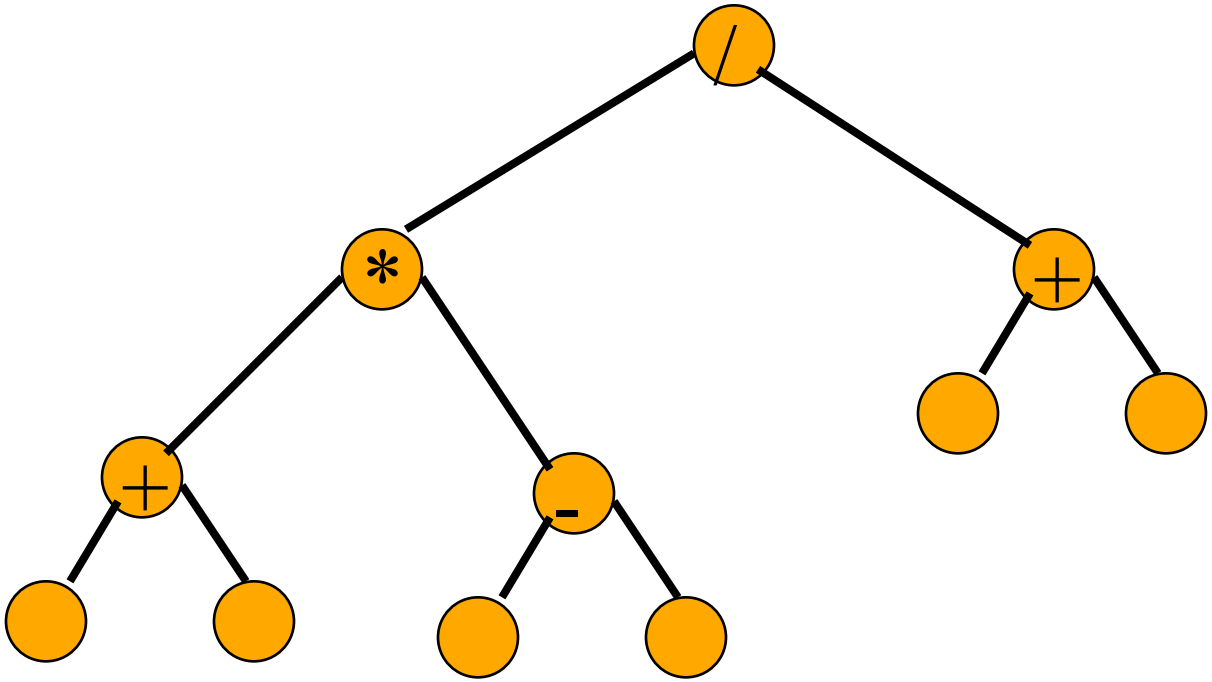

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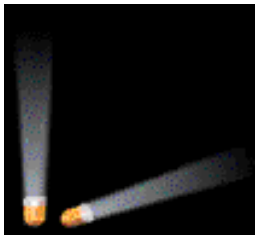
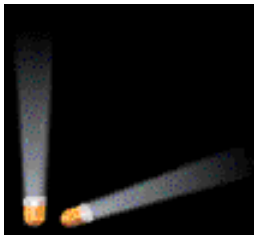
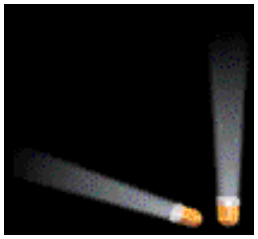
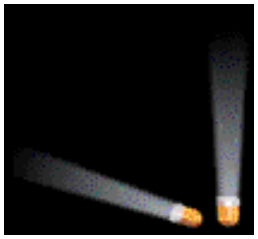
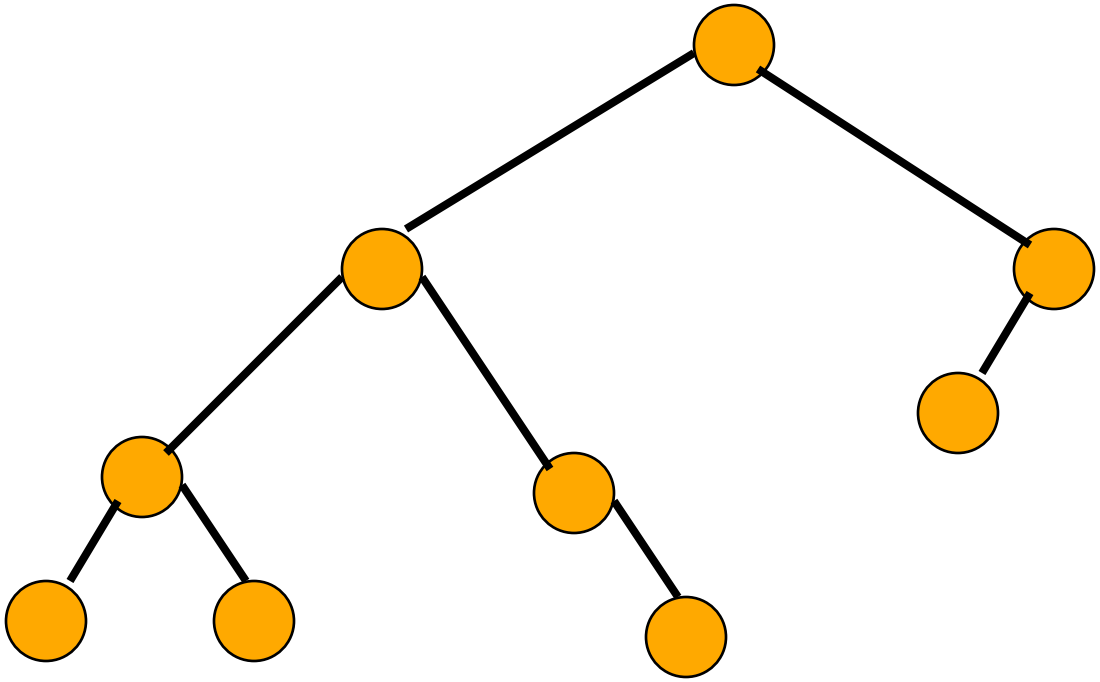
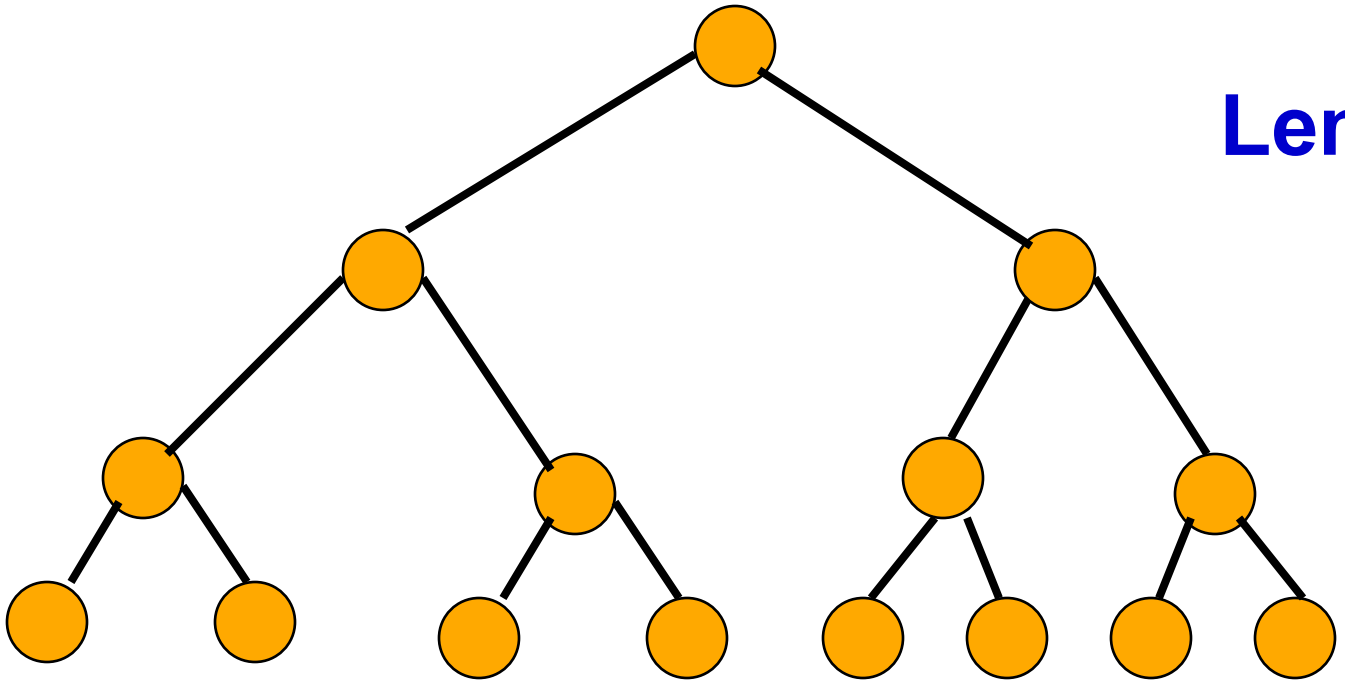







Lemma 5.2



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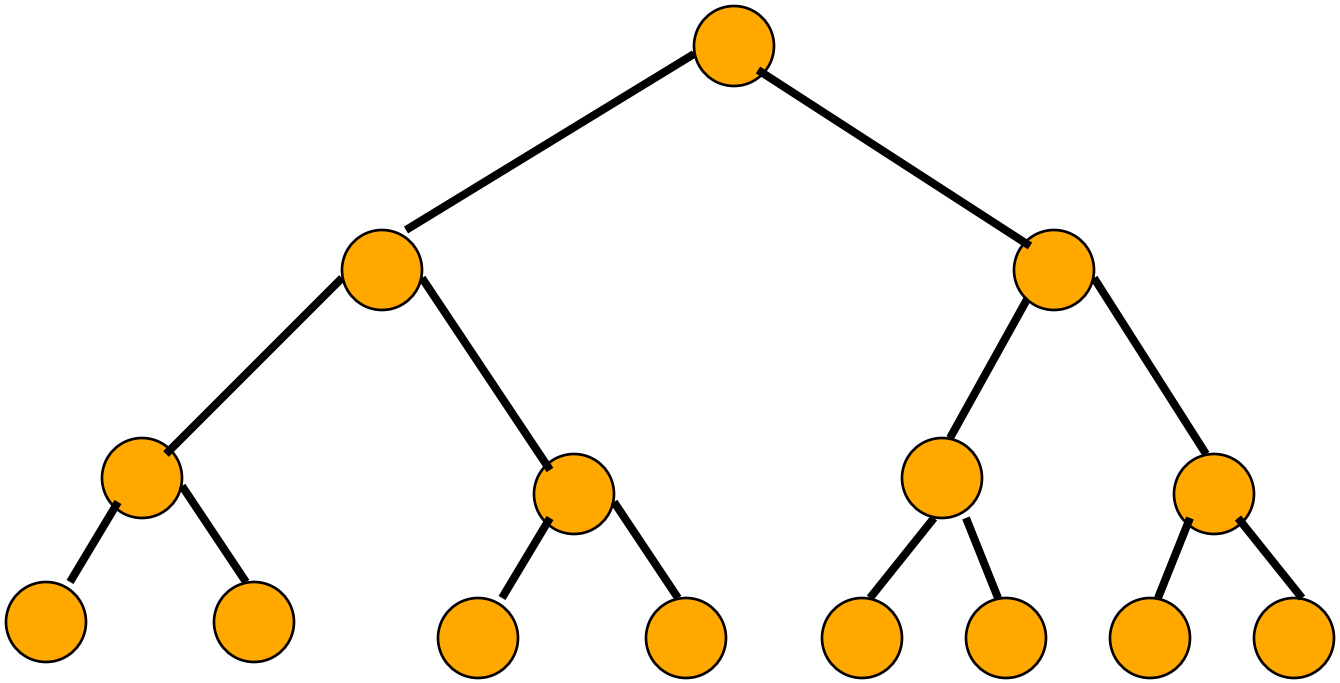

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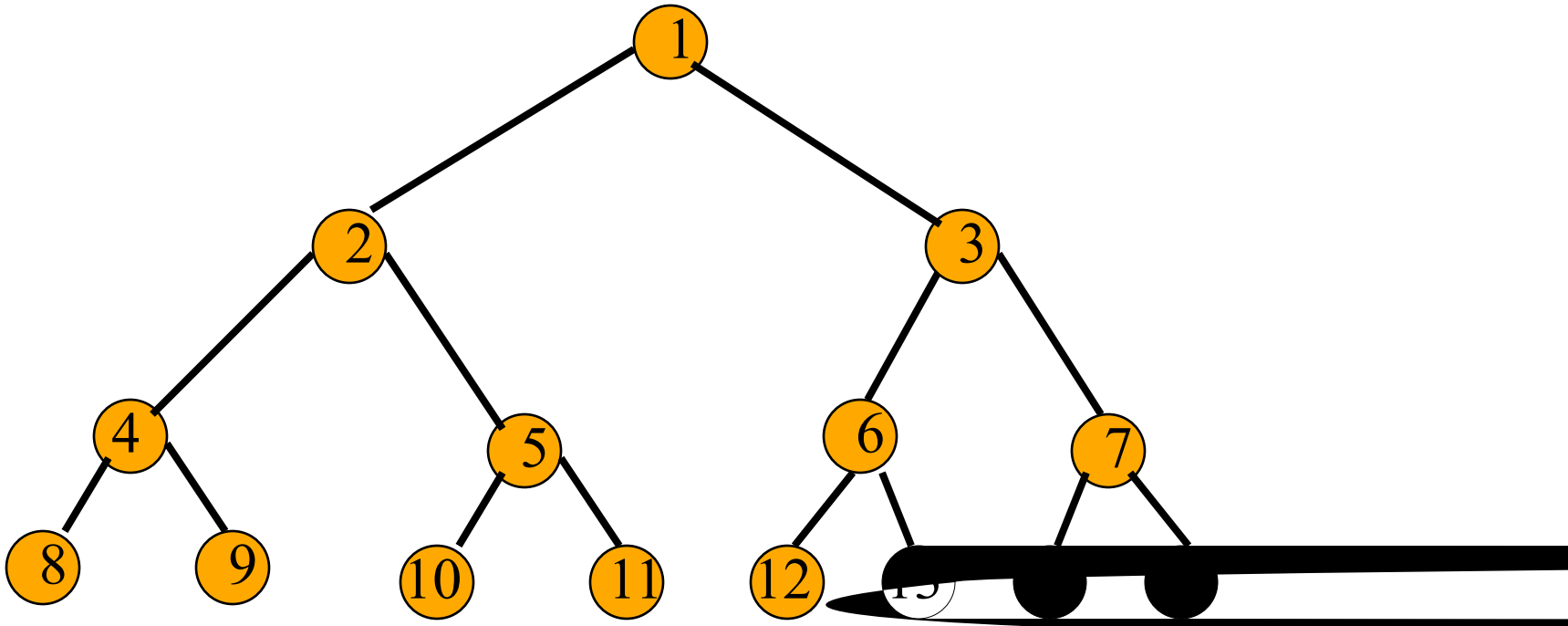

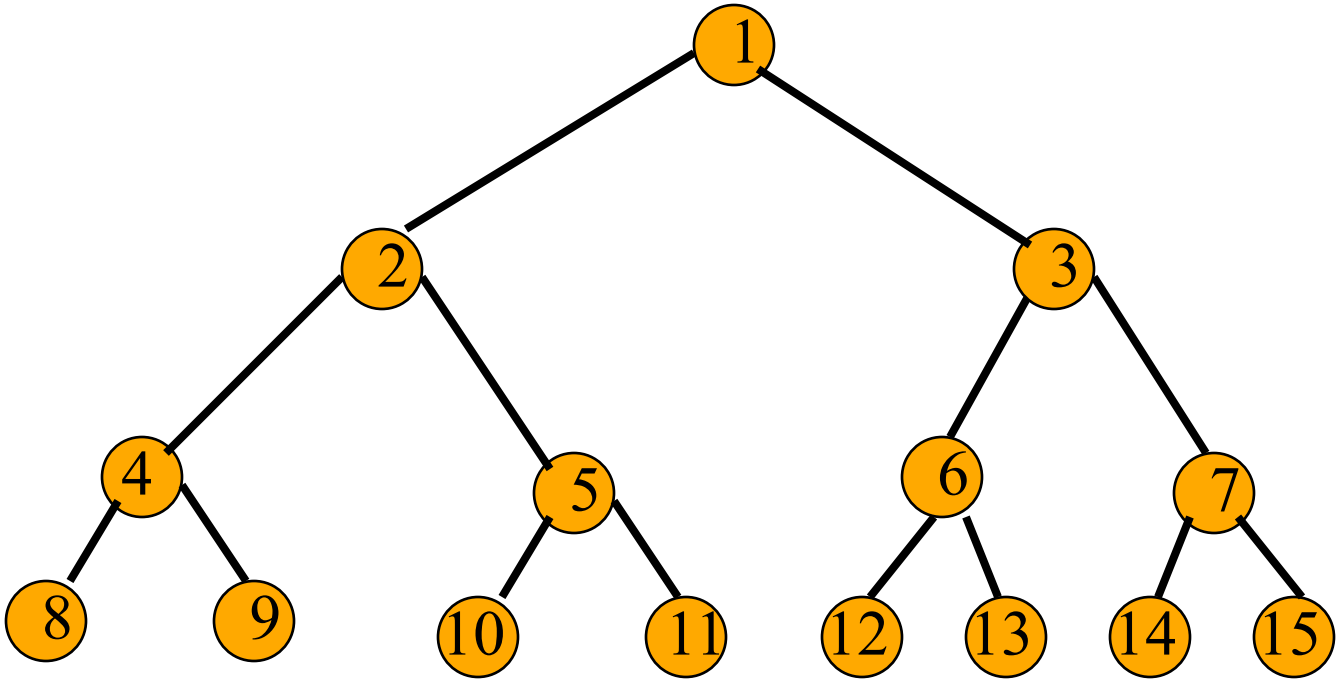

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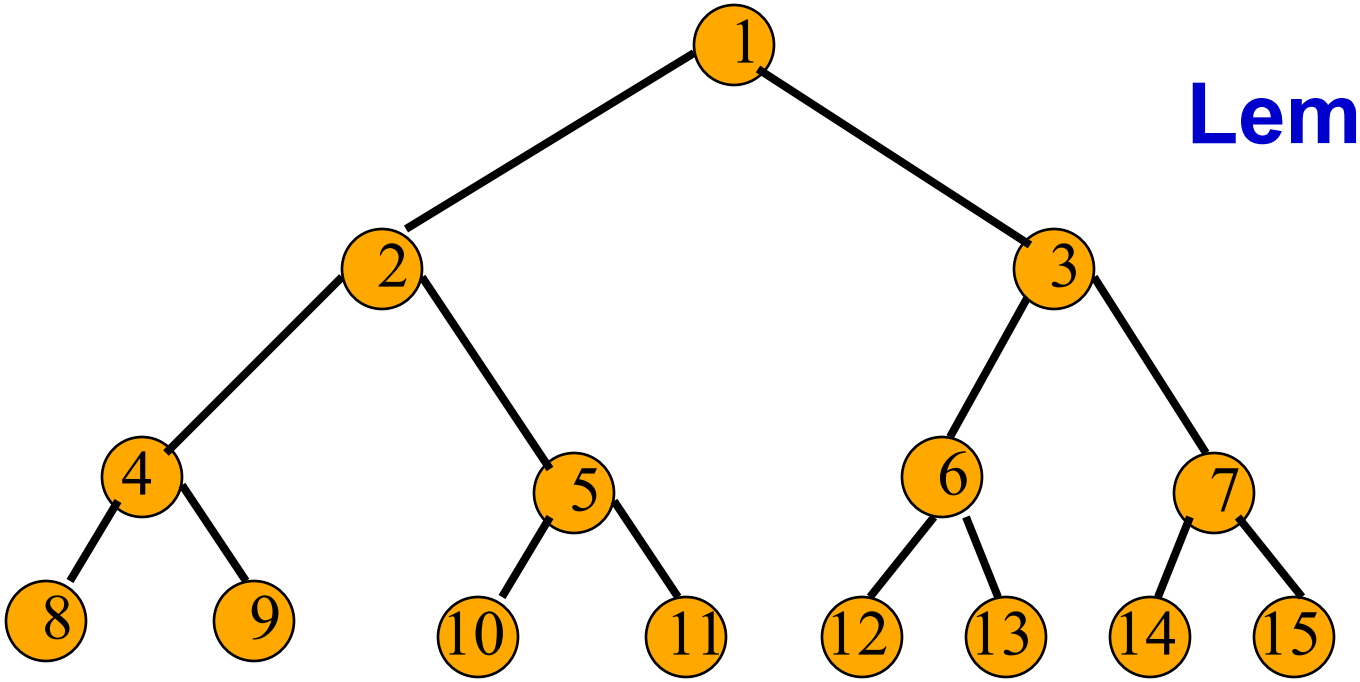

Lemma 5.4

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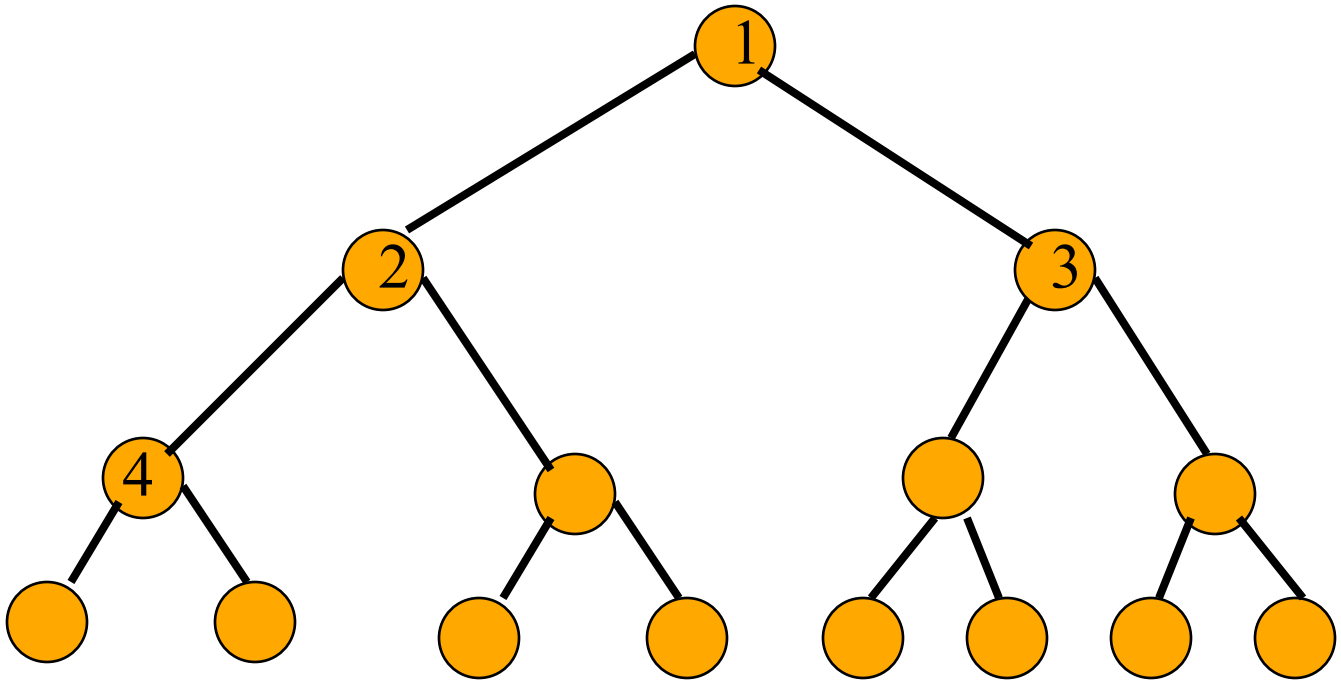
Definition: a binary tree with n nodes and depth k is **complete** iff its nodes corresponding to the nodes numbered from 1 to n in the full binary tree of depth k.



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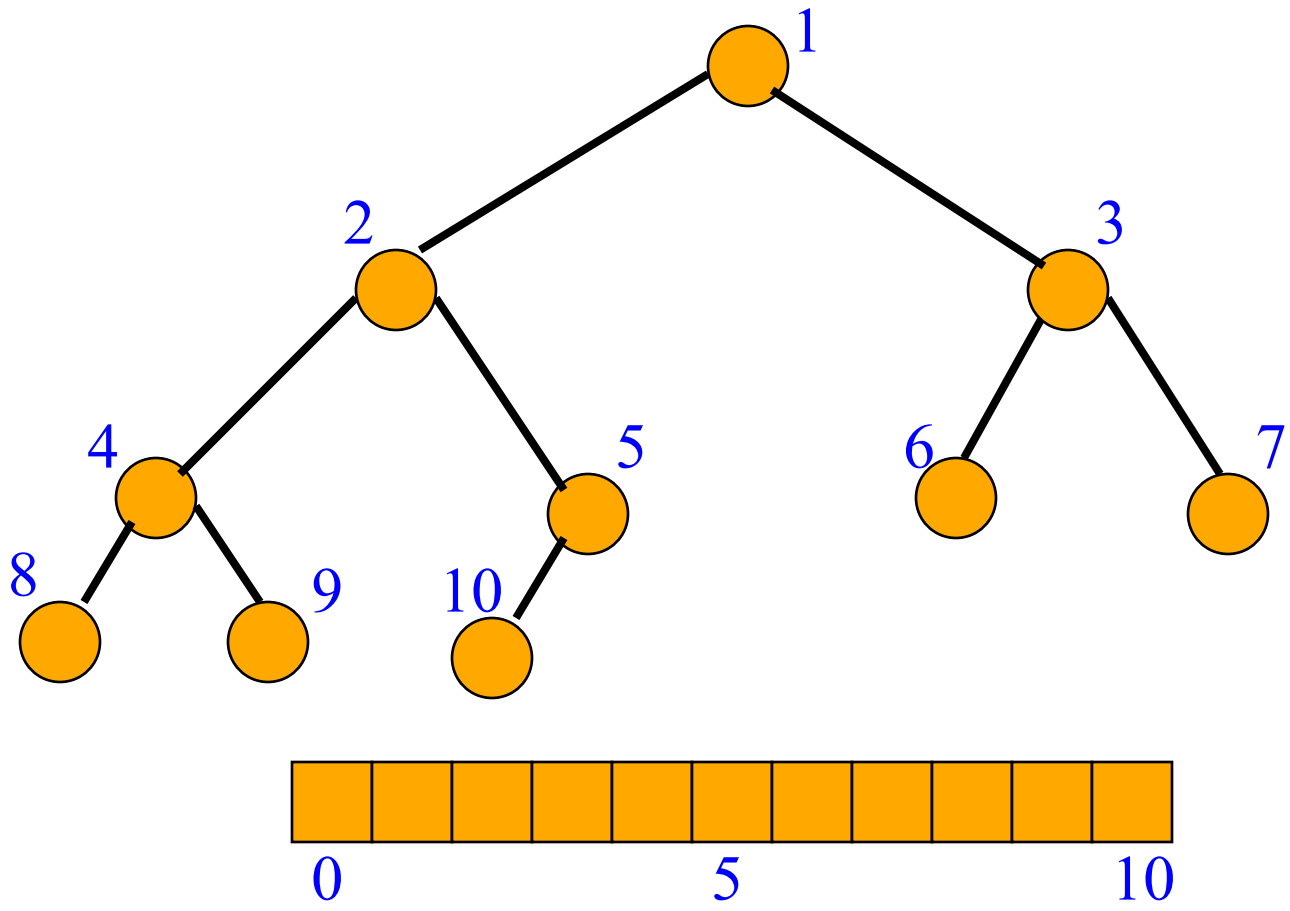
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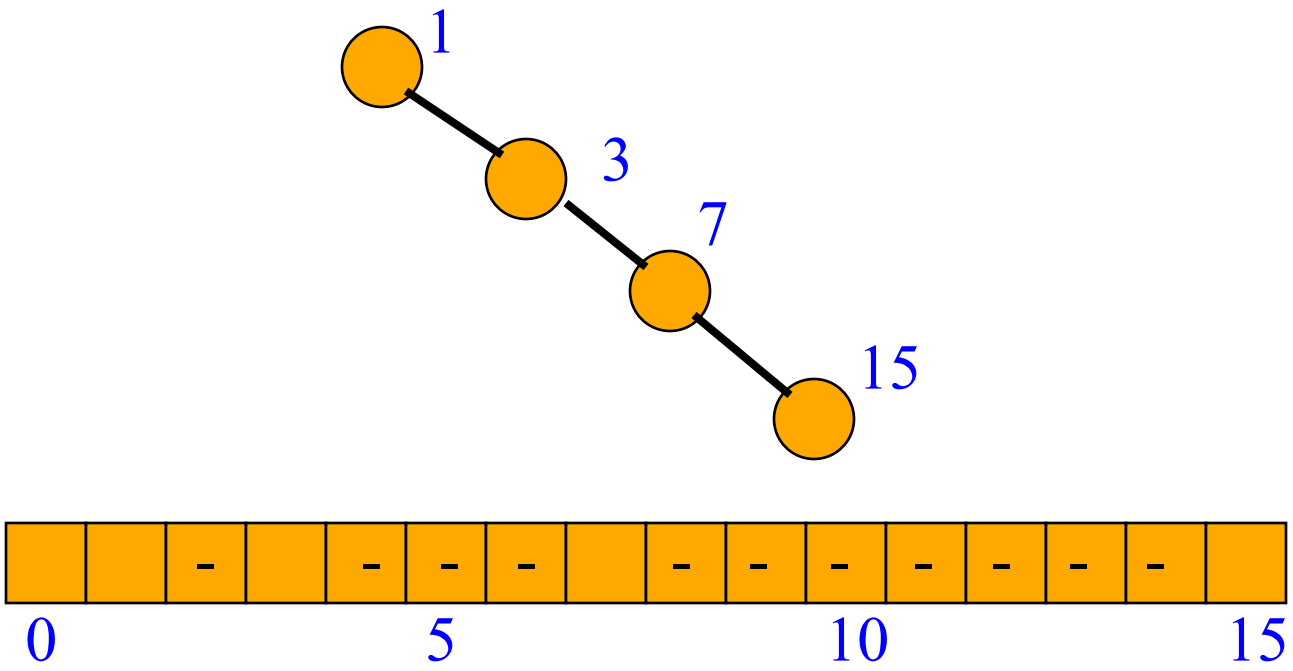

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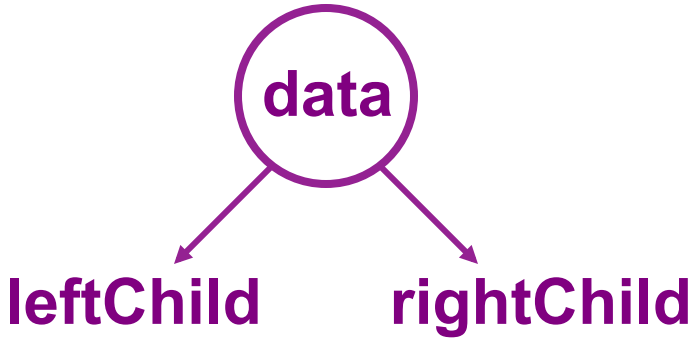
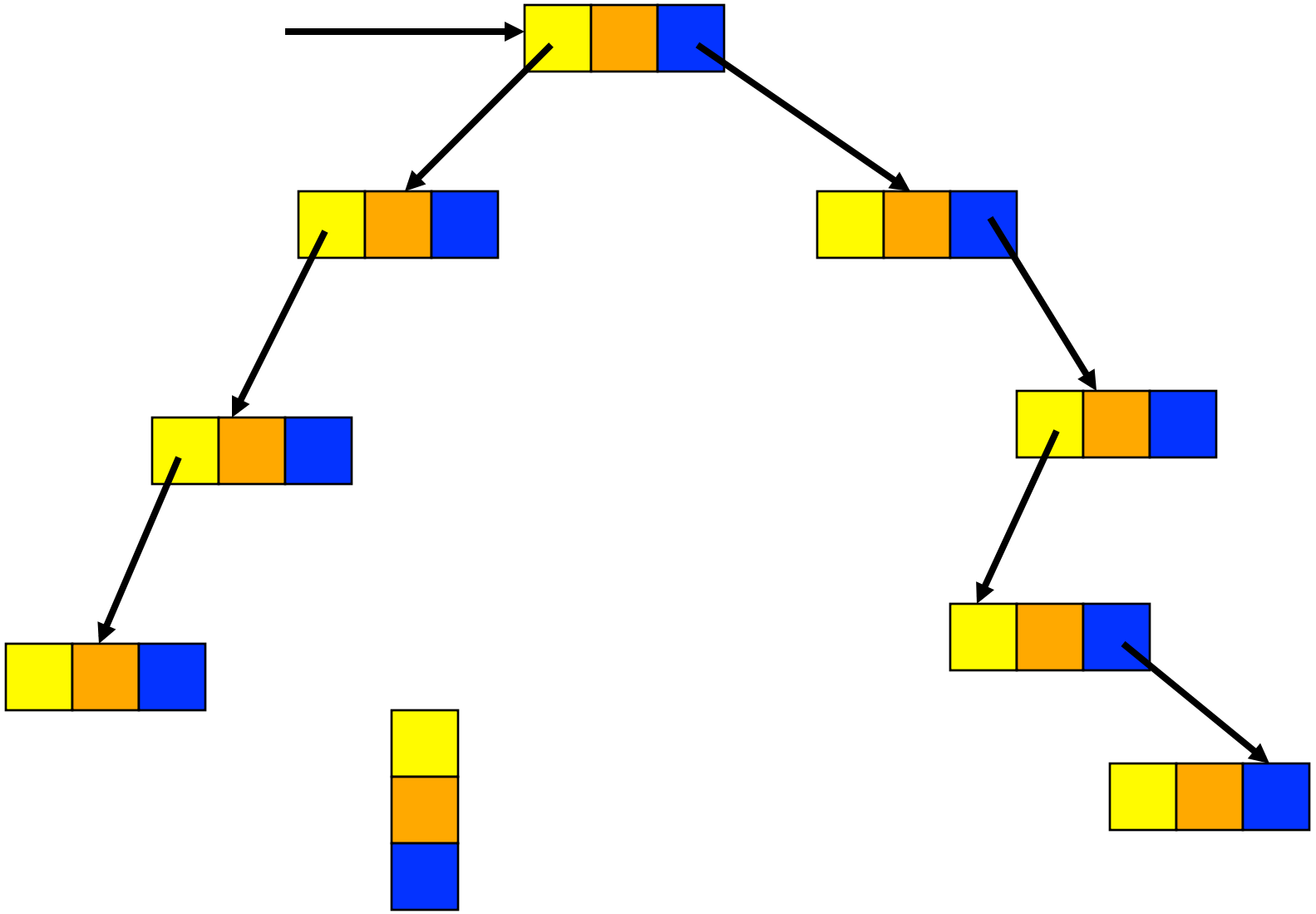
```
template <class T> class Tree;
class TreeNode {
friend class Tree<T>;
public:
    TreeNode (T& e, TreeNode<T>* left, TreeNode<T>* right)
        {data=e; leftChild=left; rightChild=right;}
private:
    T data;
    TreeNode<Y>* leftChild;
    TreeNode<Y>* rightChild;
};
```

leftChild	data	rightChild




```
template <class T>
class Tree {
public:
    // Tree operations
…
private:
    TreeNode<T>* root;
};
```

If necessary, a 4th field, parent, may be included in the node.



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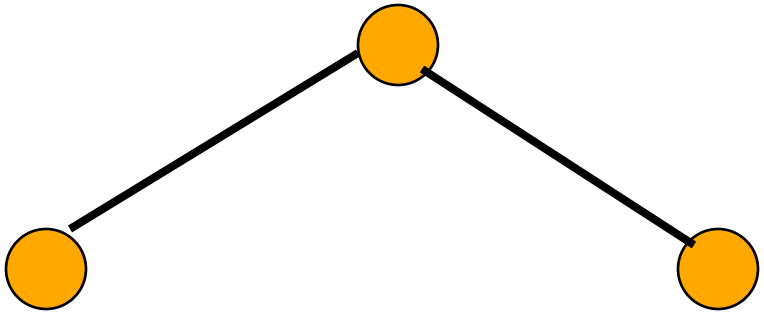

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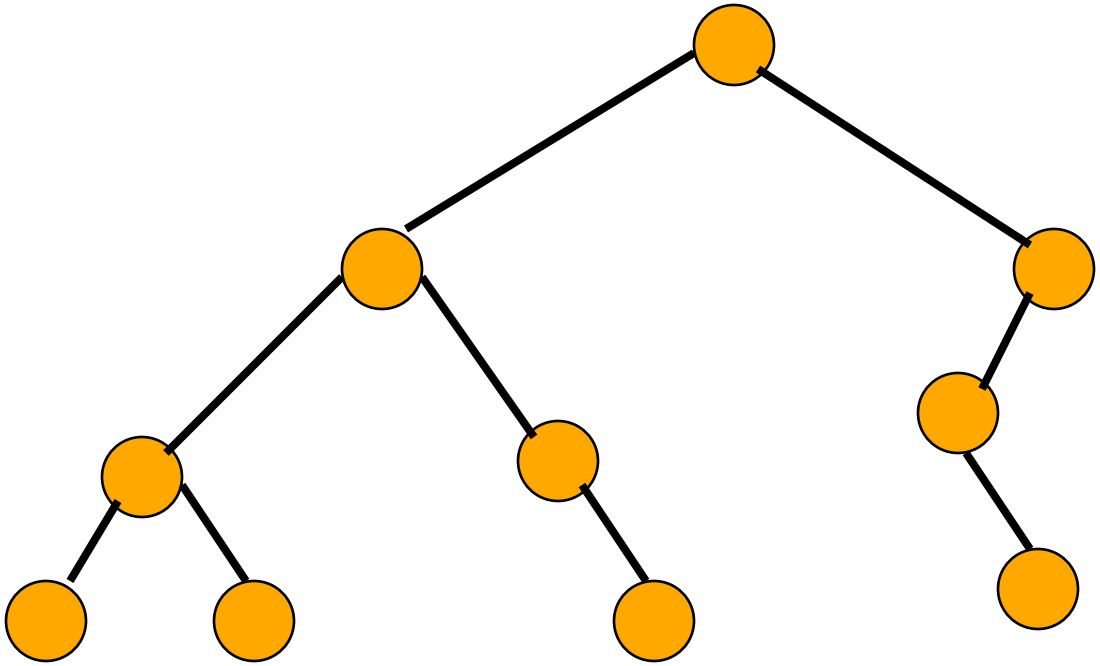

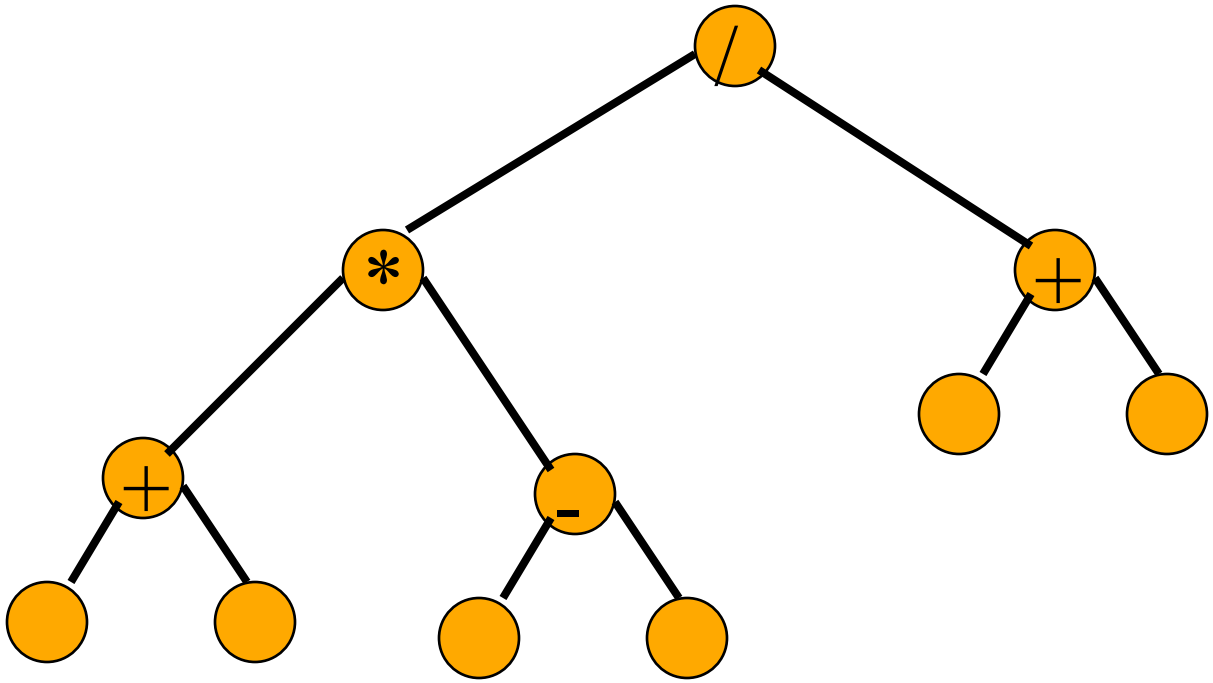

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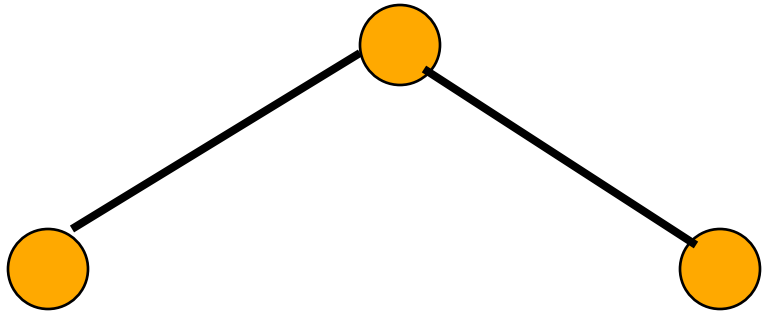

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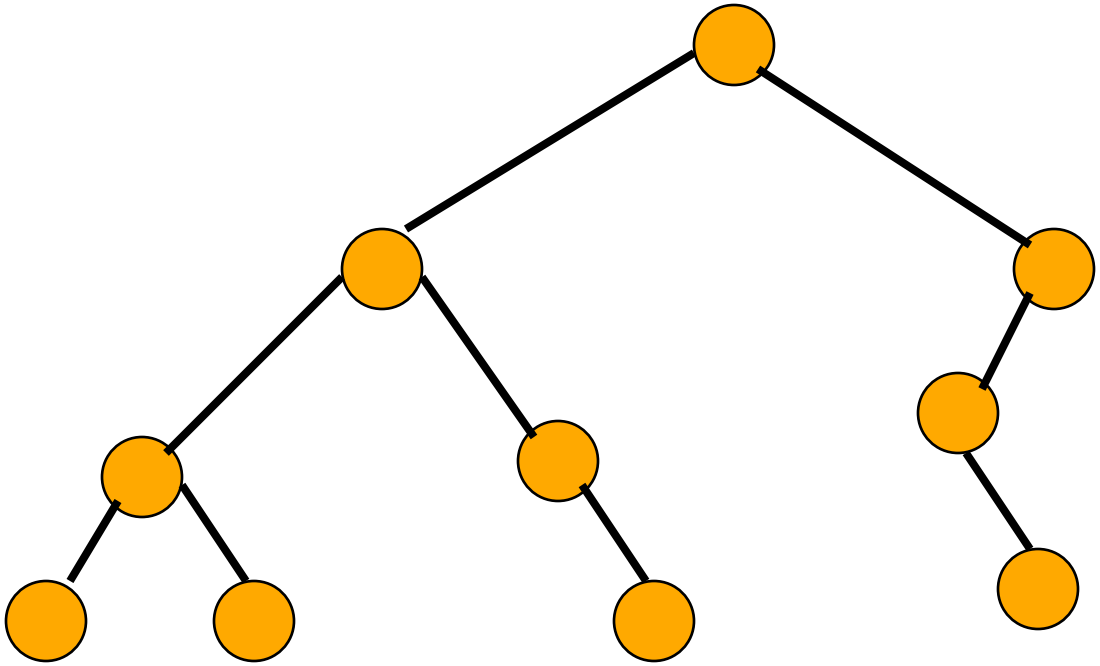

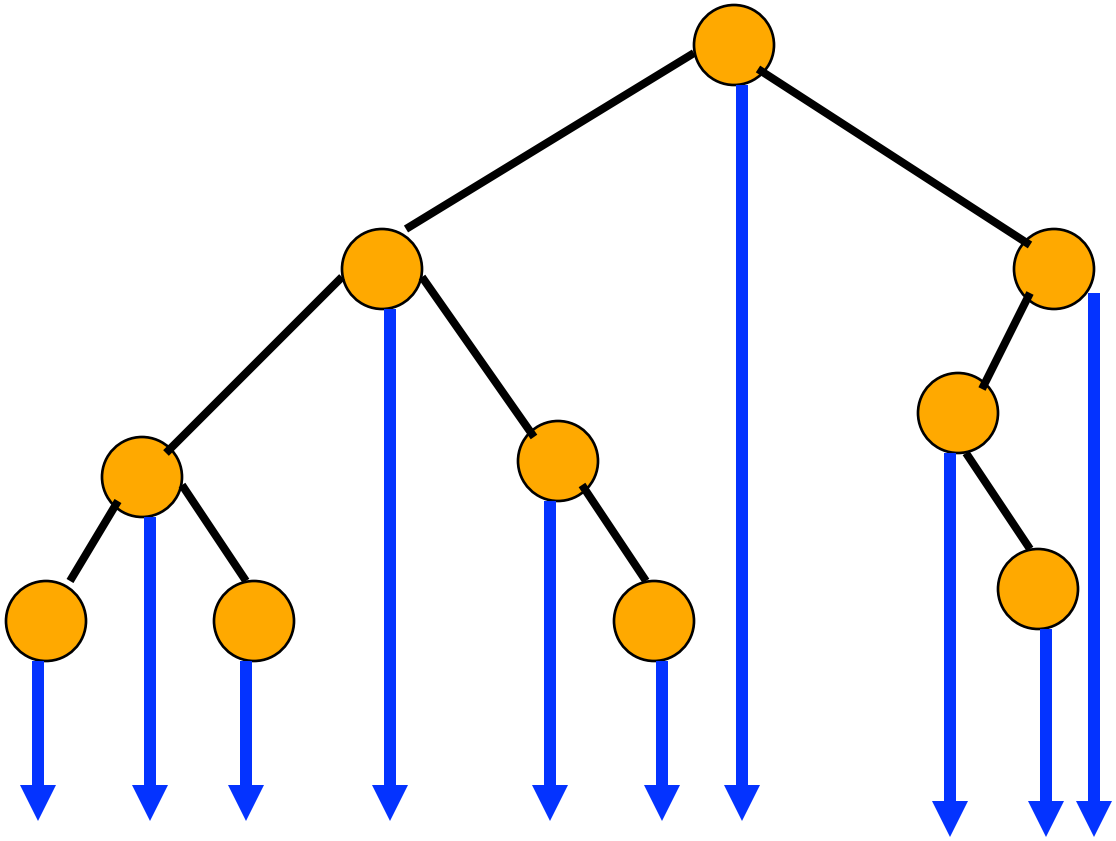

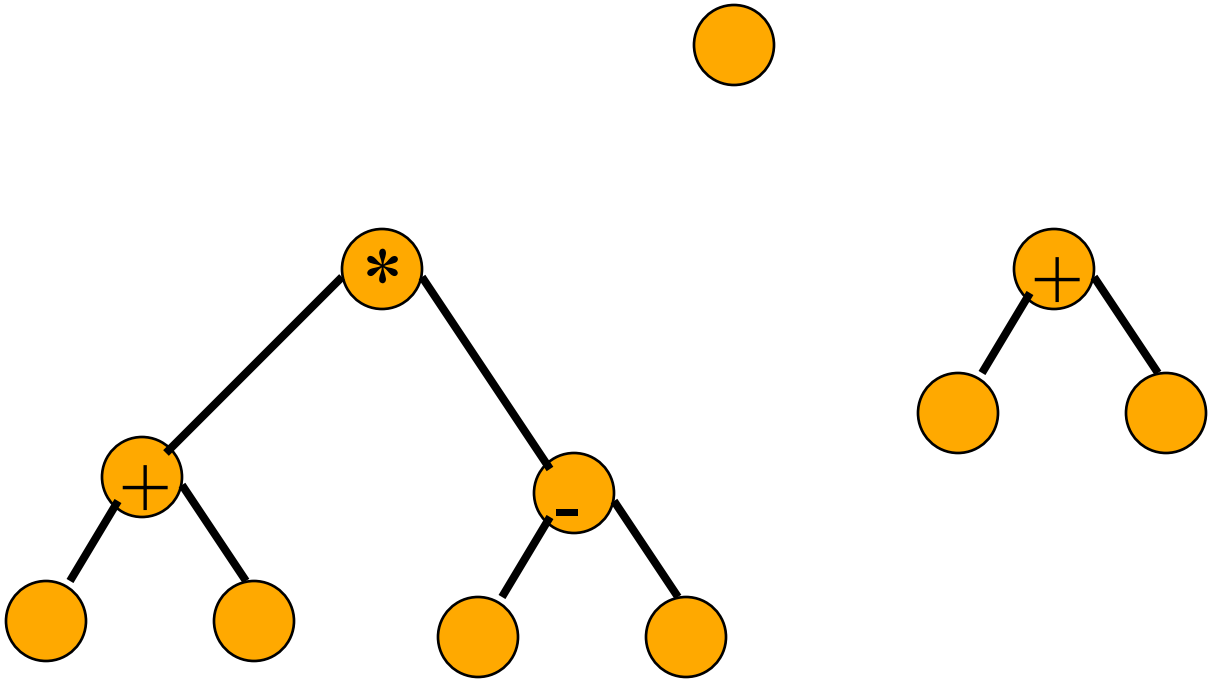

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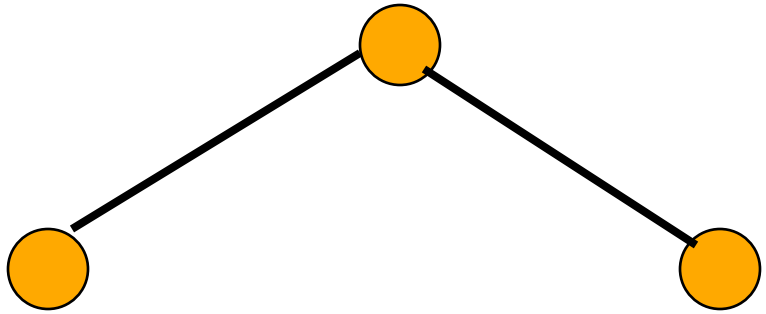

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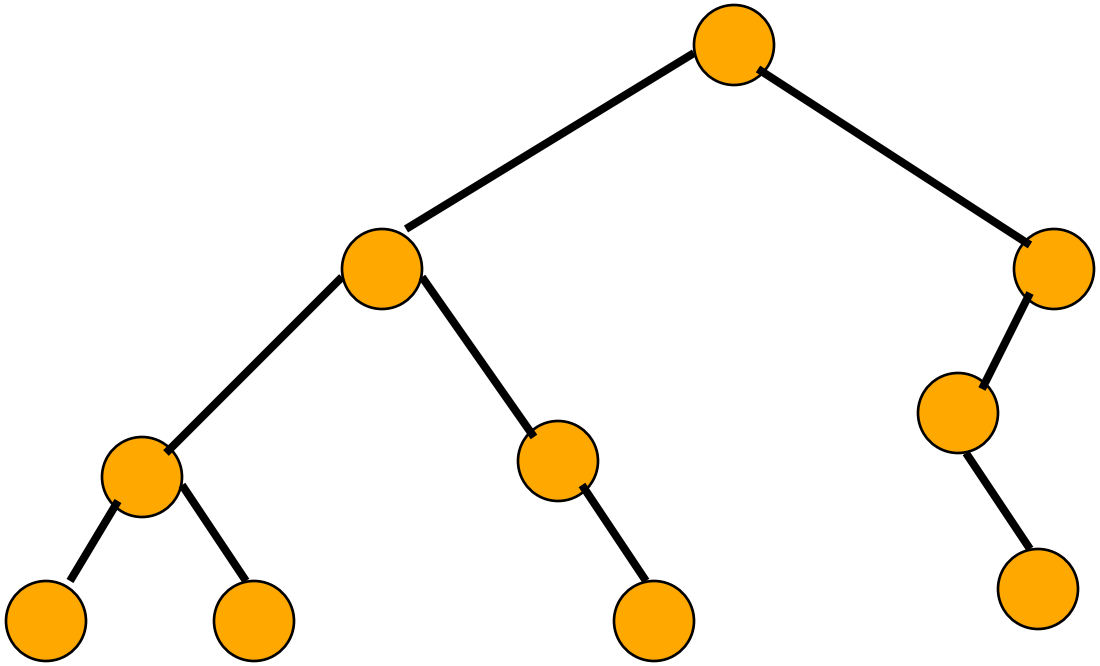

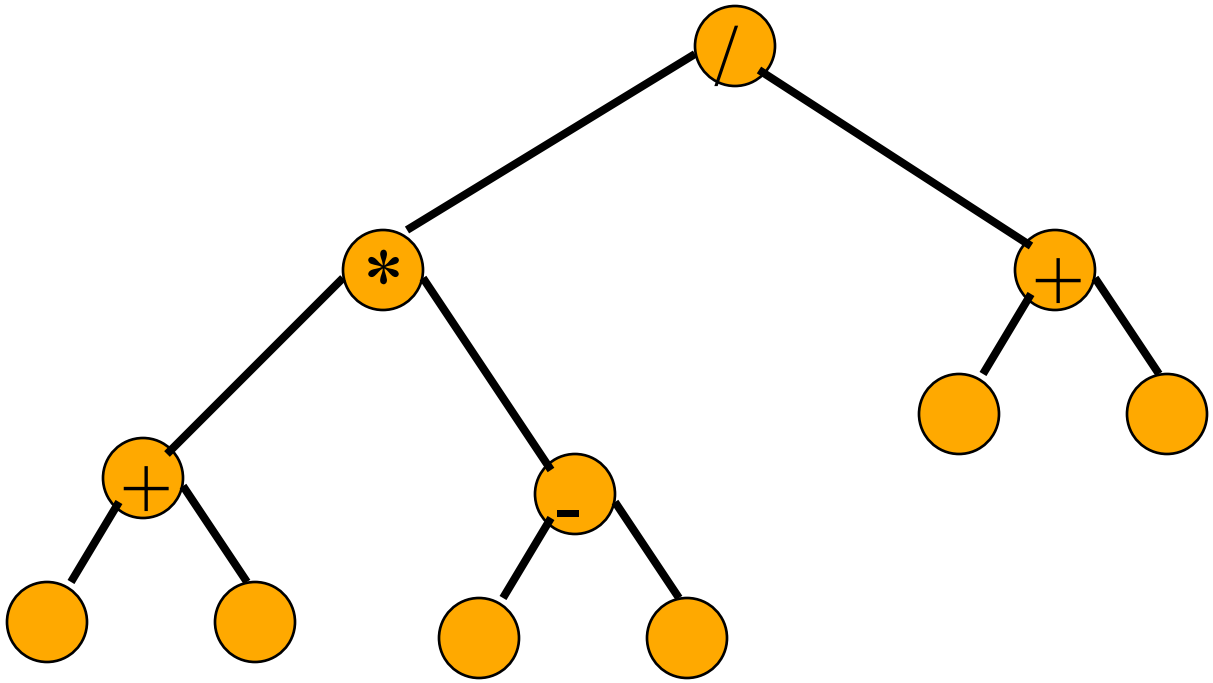

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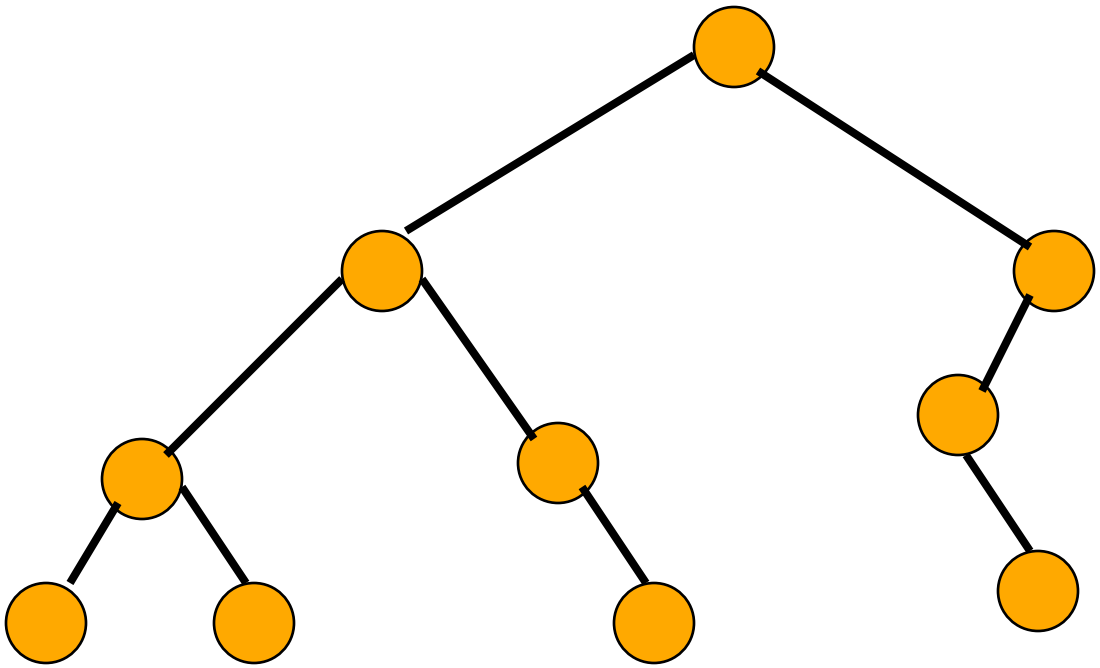

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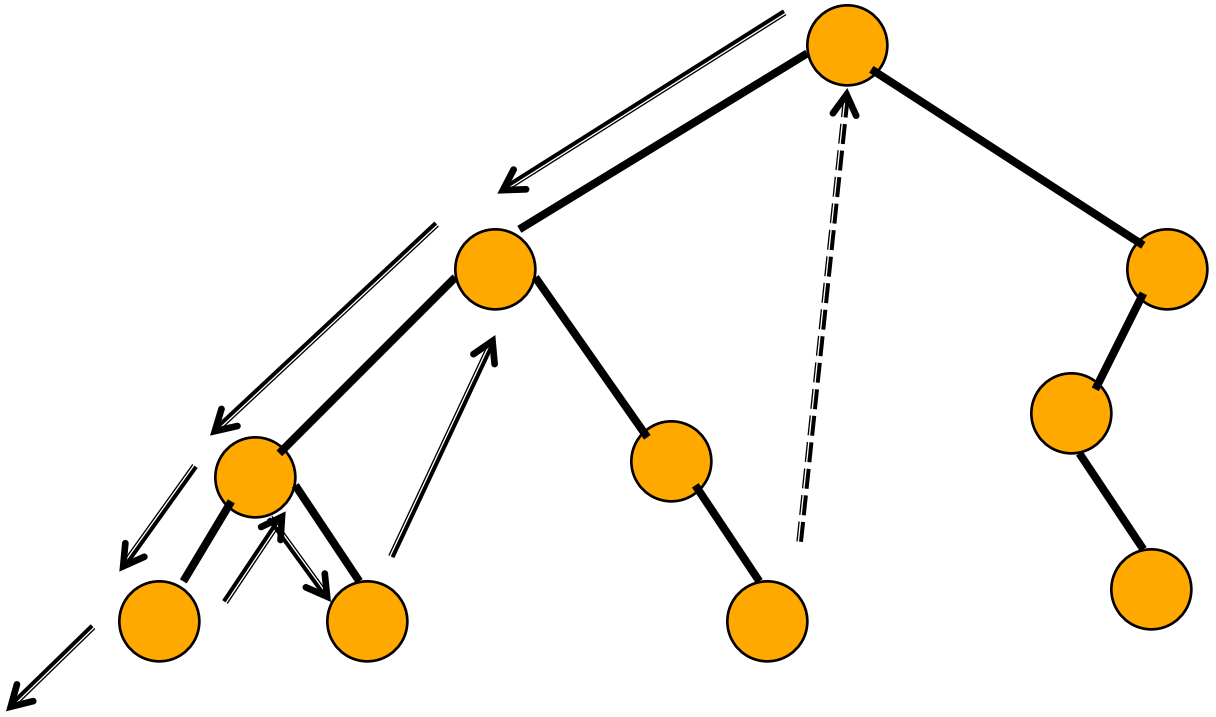
Iterative Inorder Traversal



```
1 template <class T>
2 void Tree<T>::NonrecInorder()
3 { // Nonrecursive inorder traversal using a stack
4   Stack<TreeNode<T>*> s; // declare and initialize a stack
5   TreeNode<T>* currentNode=root;
6   while (1) {
7     while (currentNode) { // move down leftChild
8       s.Push(currentNode); // add to stack
9       currentNode=currentNode→leftChild;
10    }
11    If (s.IsEmpty()) return;
12    currentNode=s.Top();
13    s.Pop(); // delete from stack
14    Visit(currentNode);
15    currentNode=currentNode→rightChild;
16  }
17}
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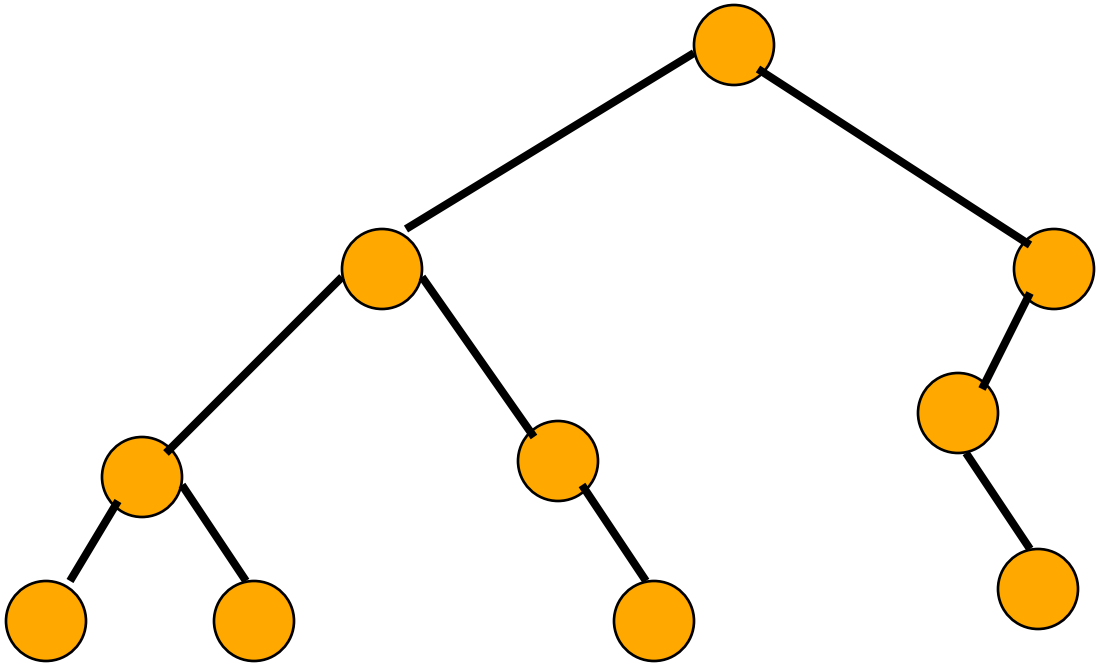

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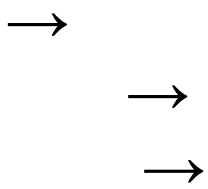

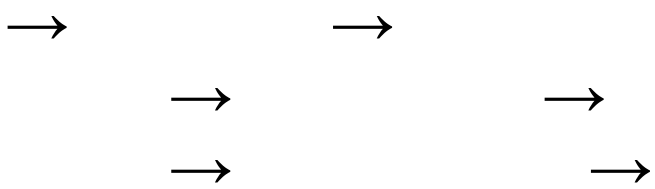

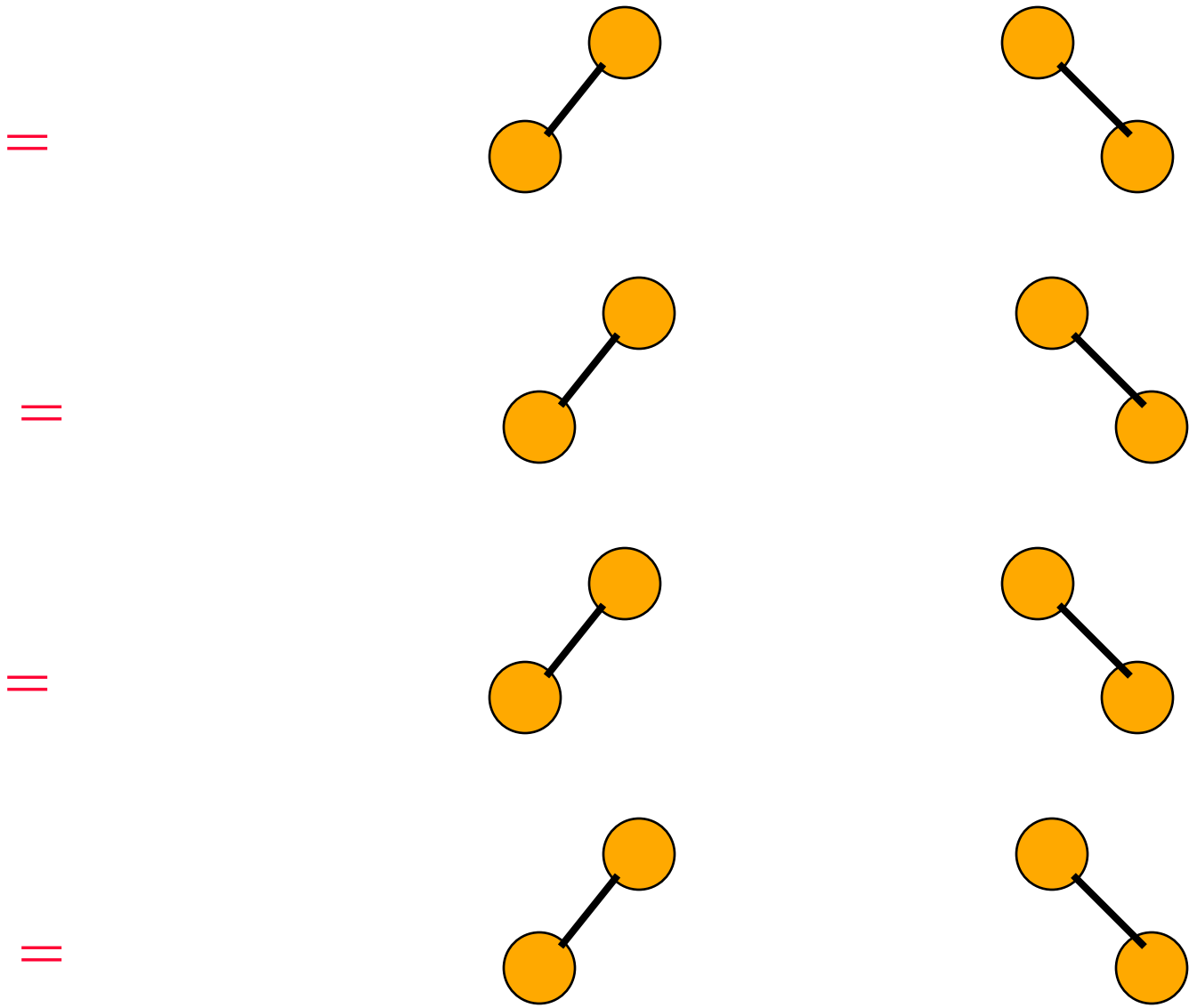

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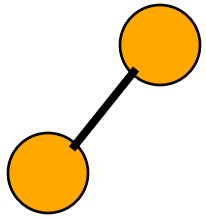

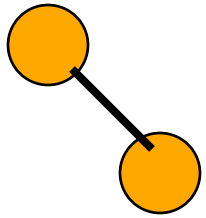

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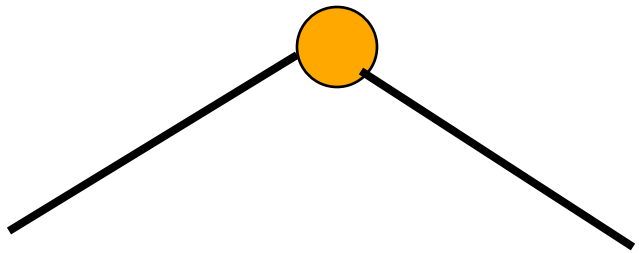

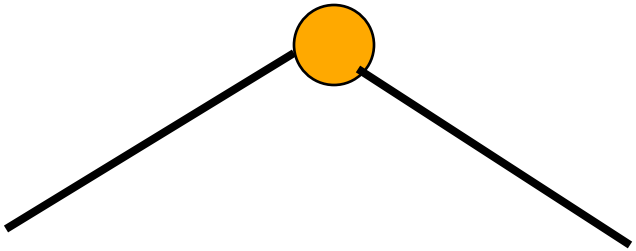

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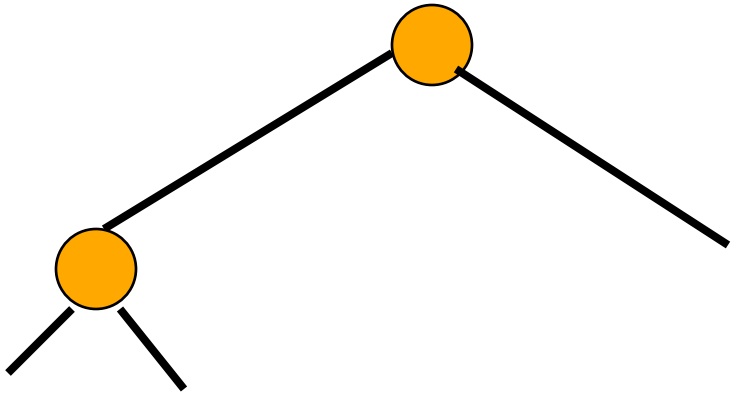

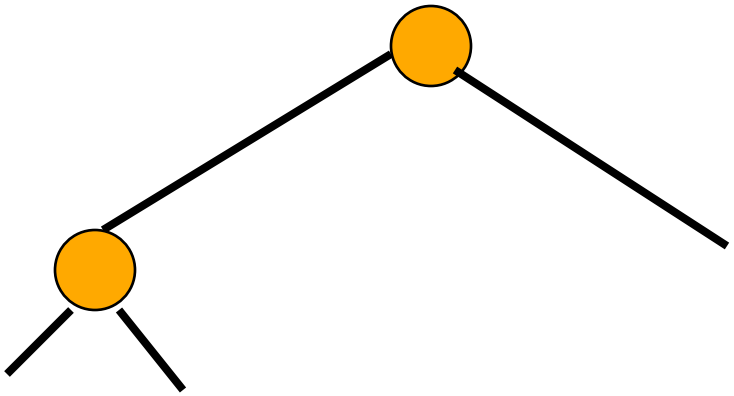

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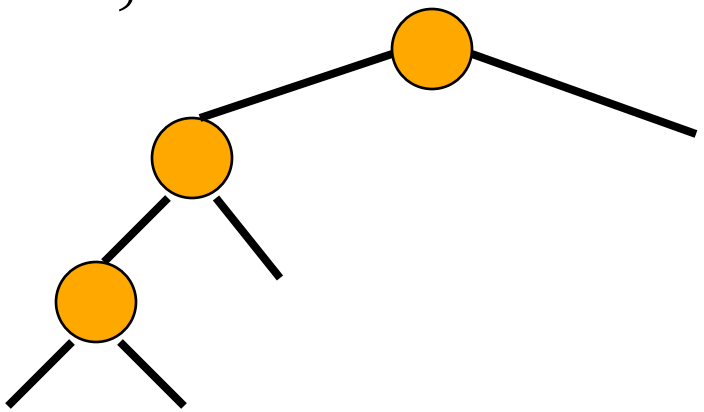

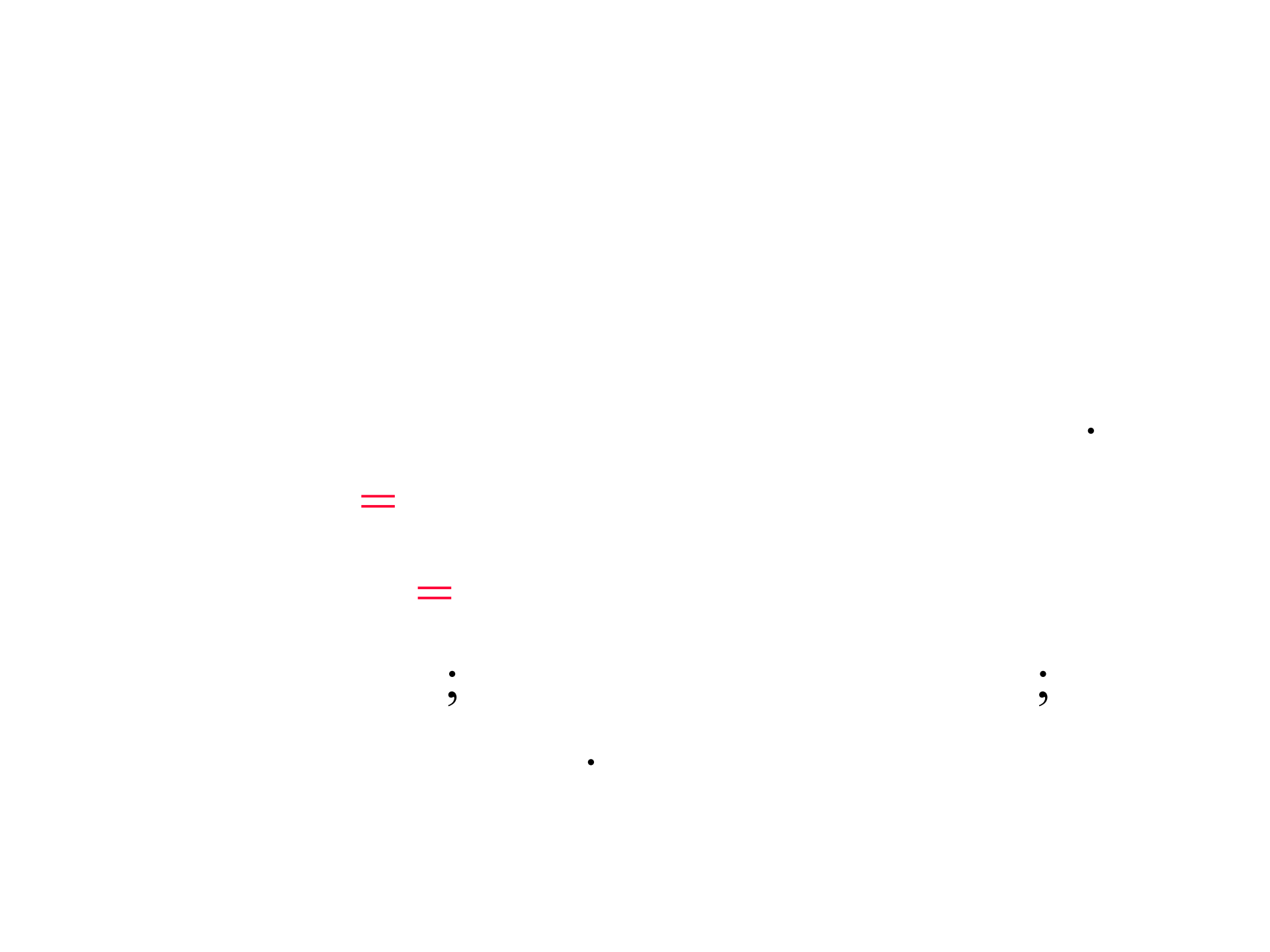

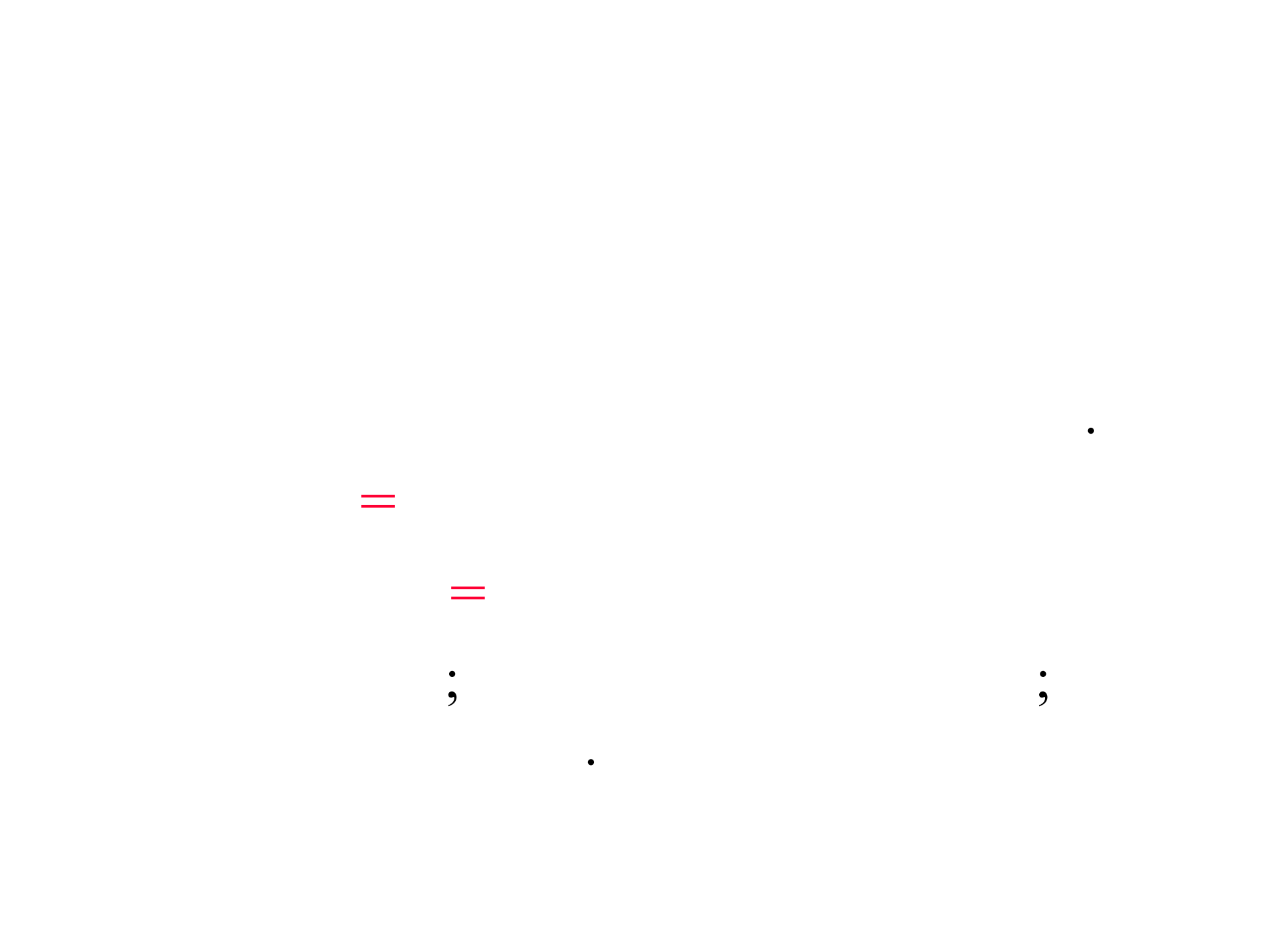

Exercises:

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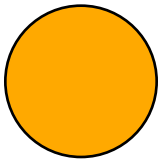

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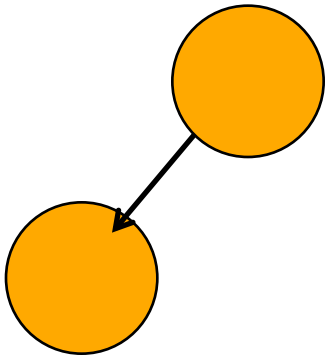

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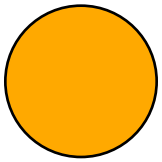

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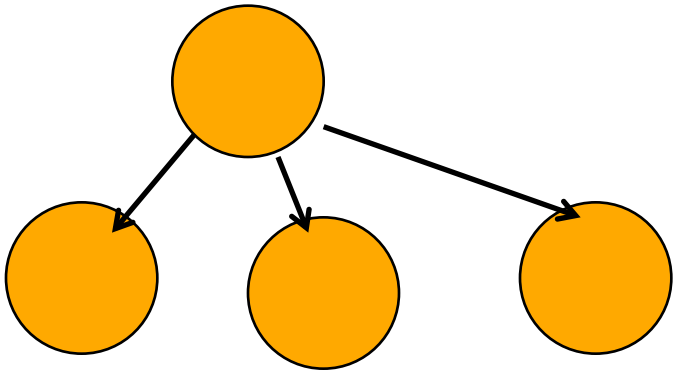

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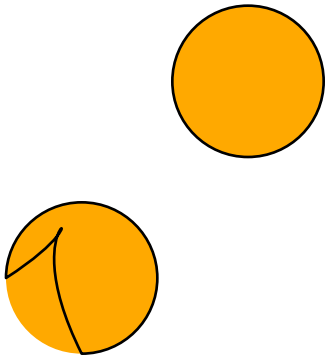

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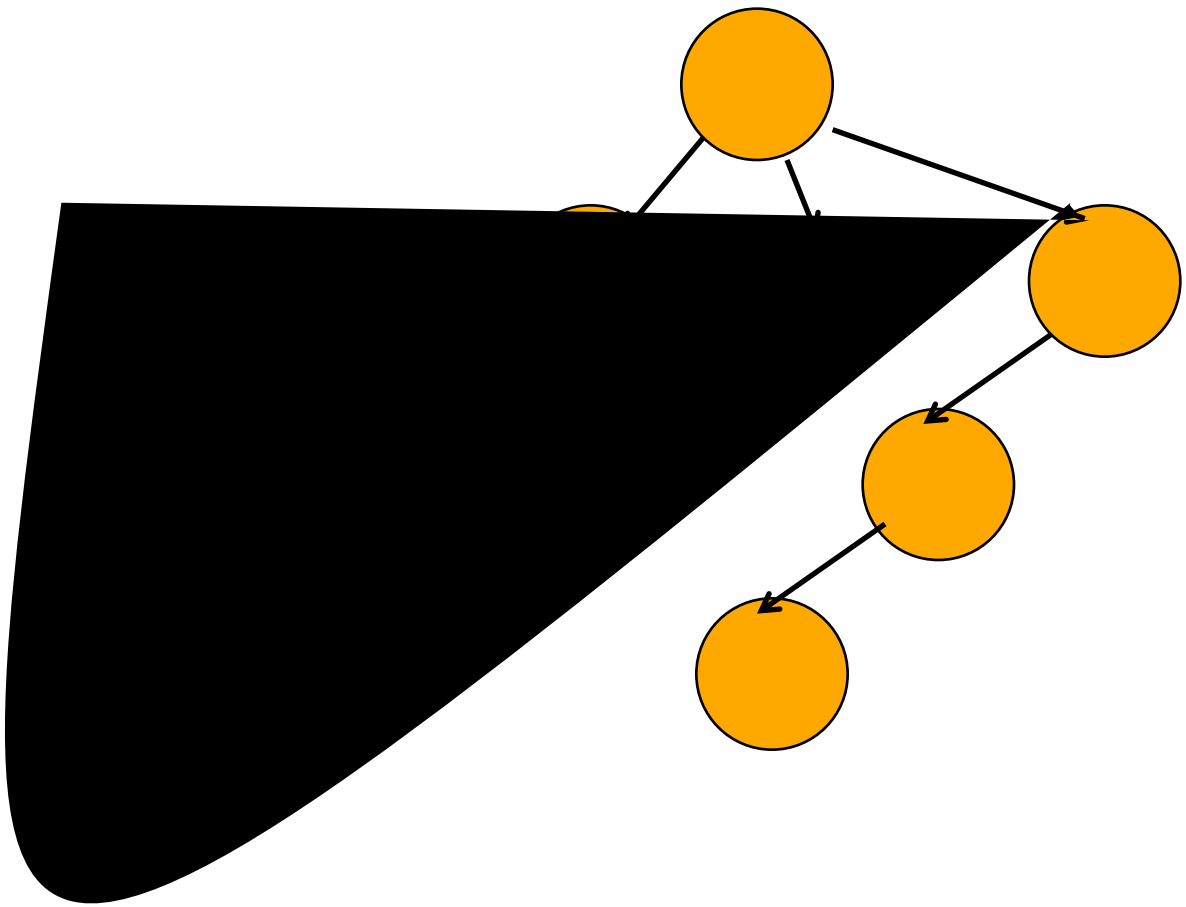

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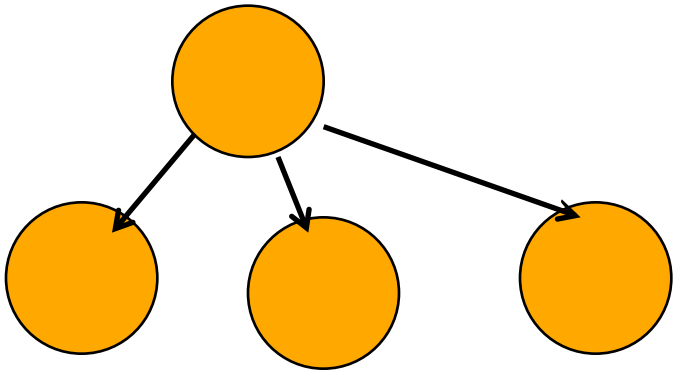

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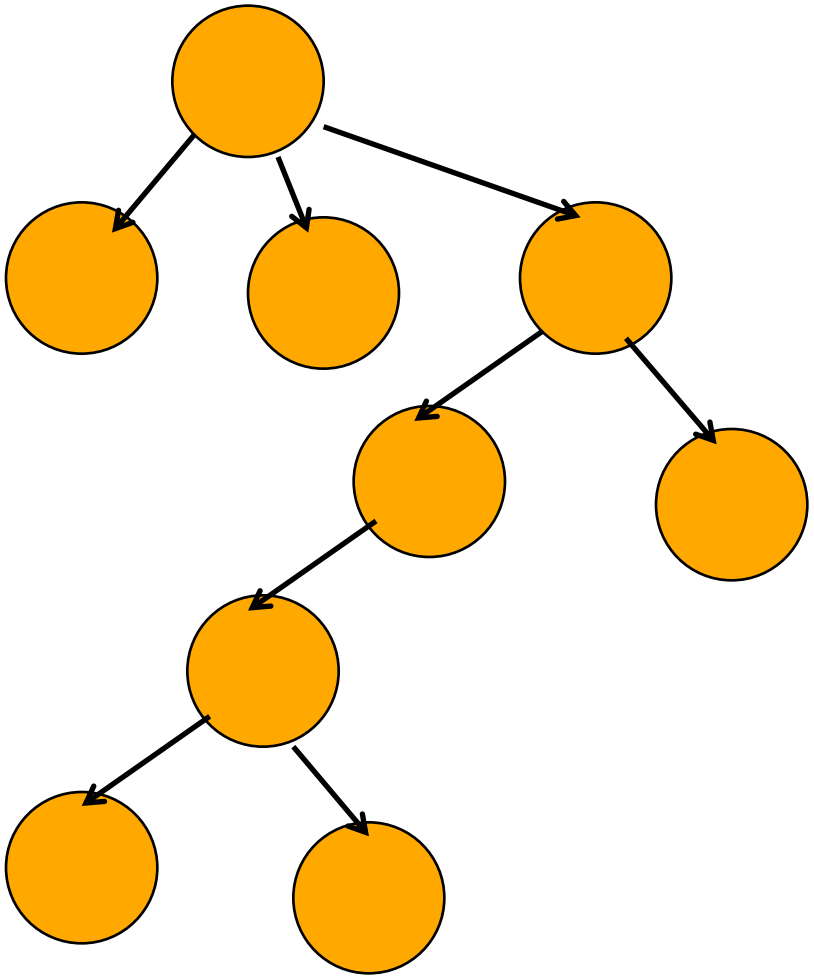

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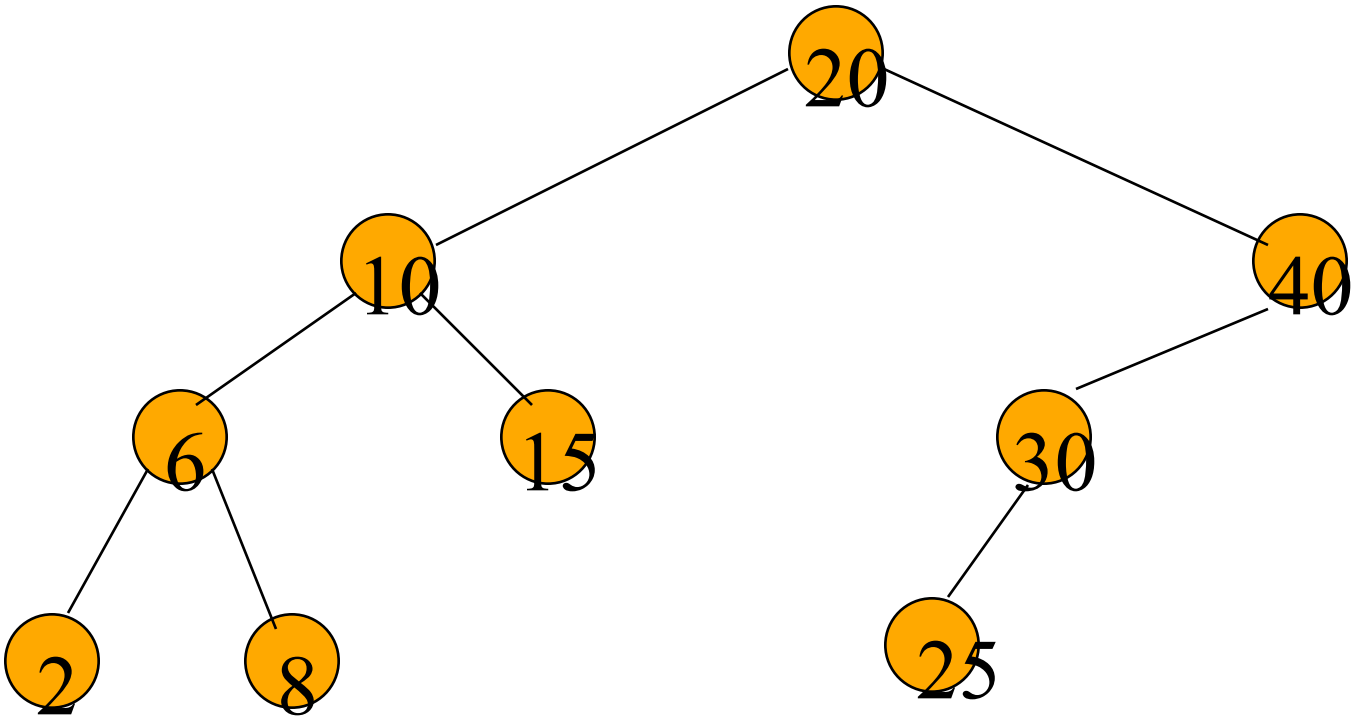
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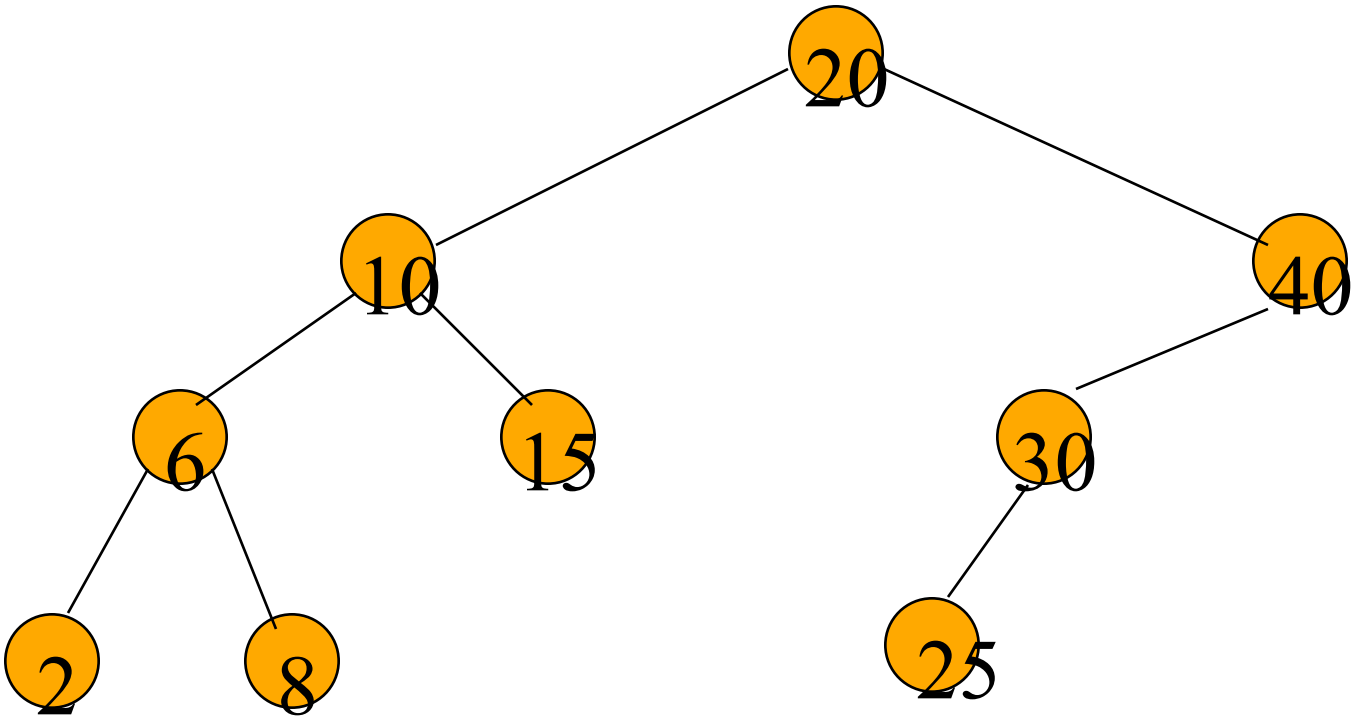
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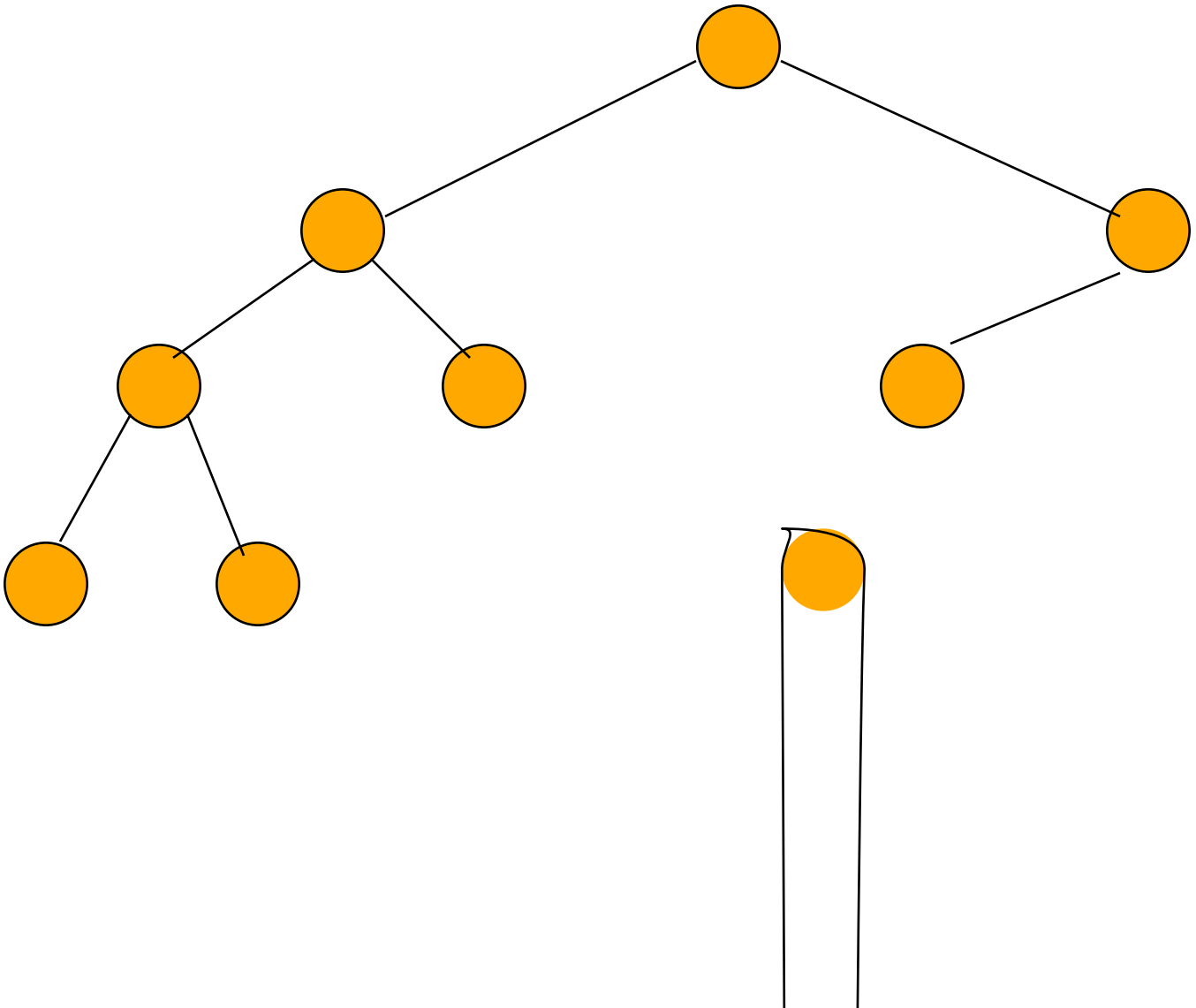

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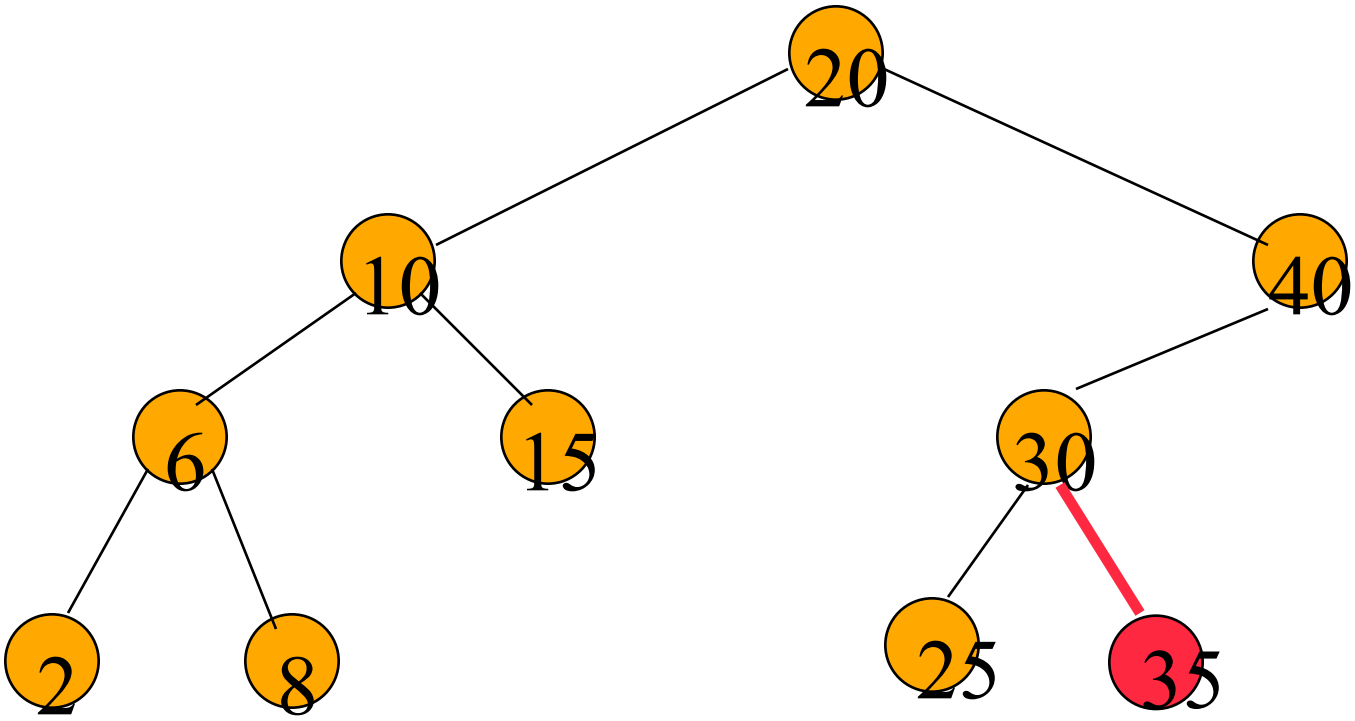
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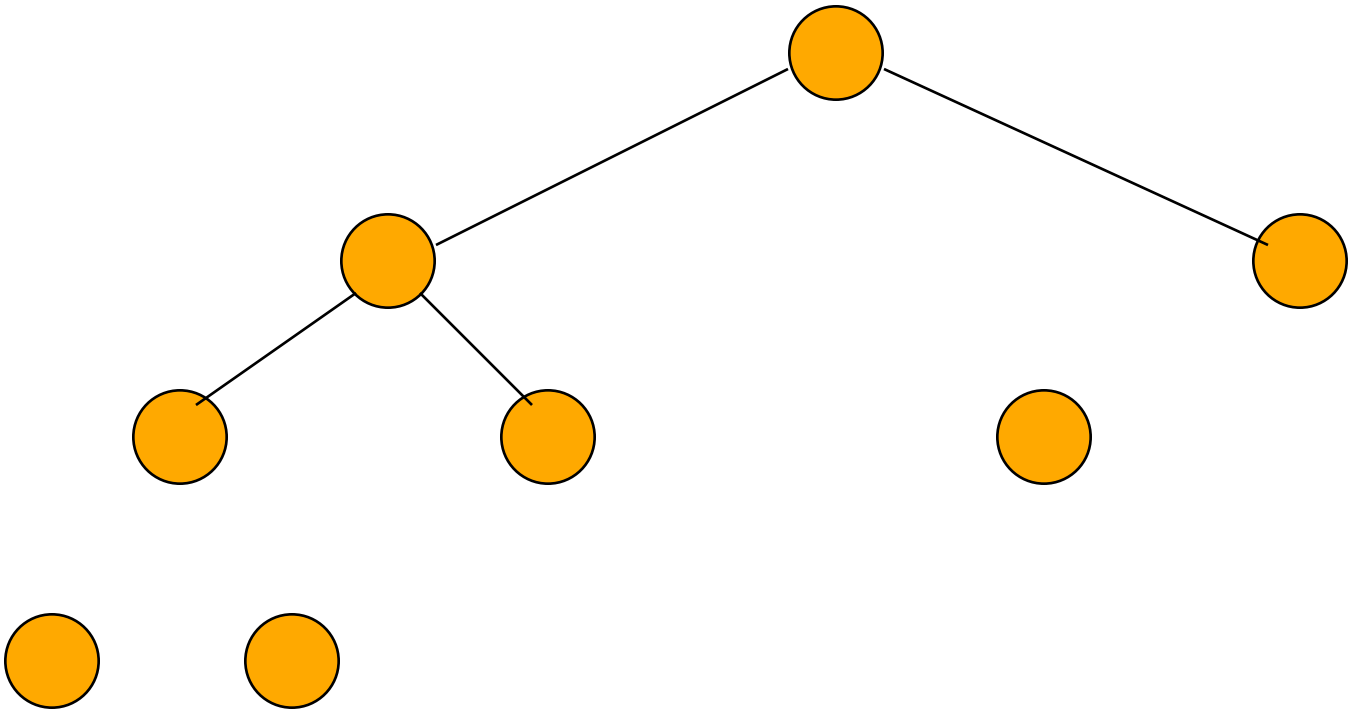

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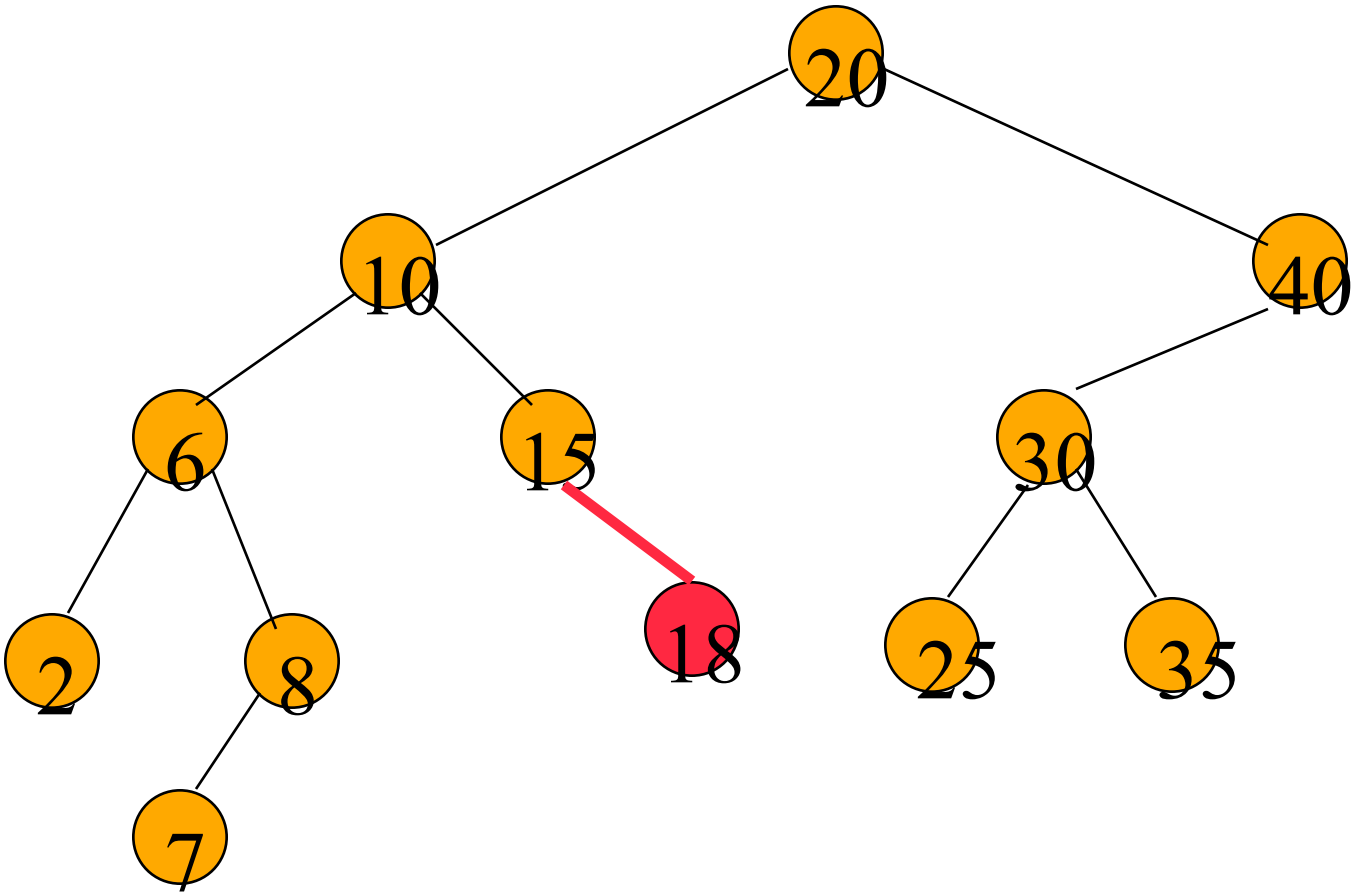
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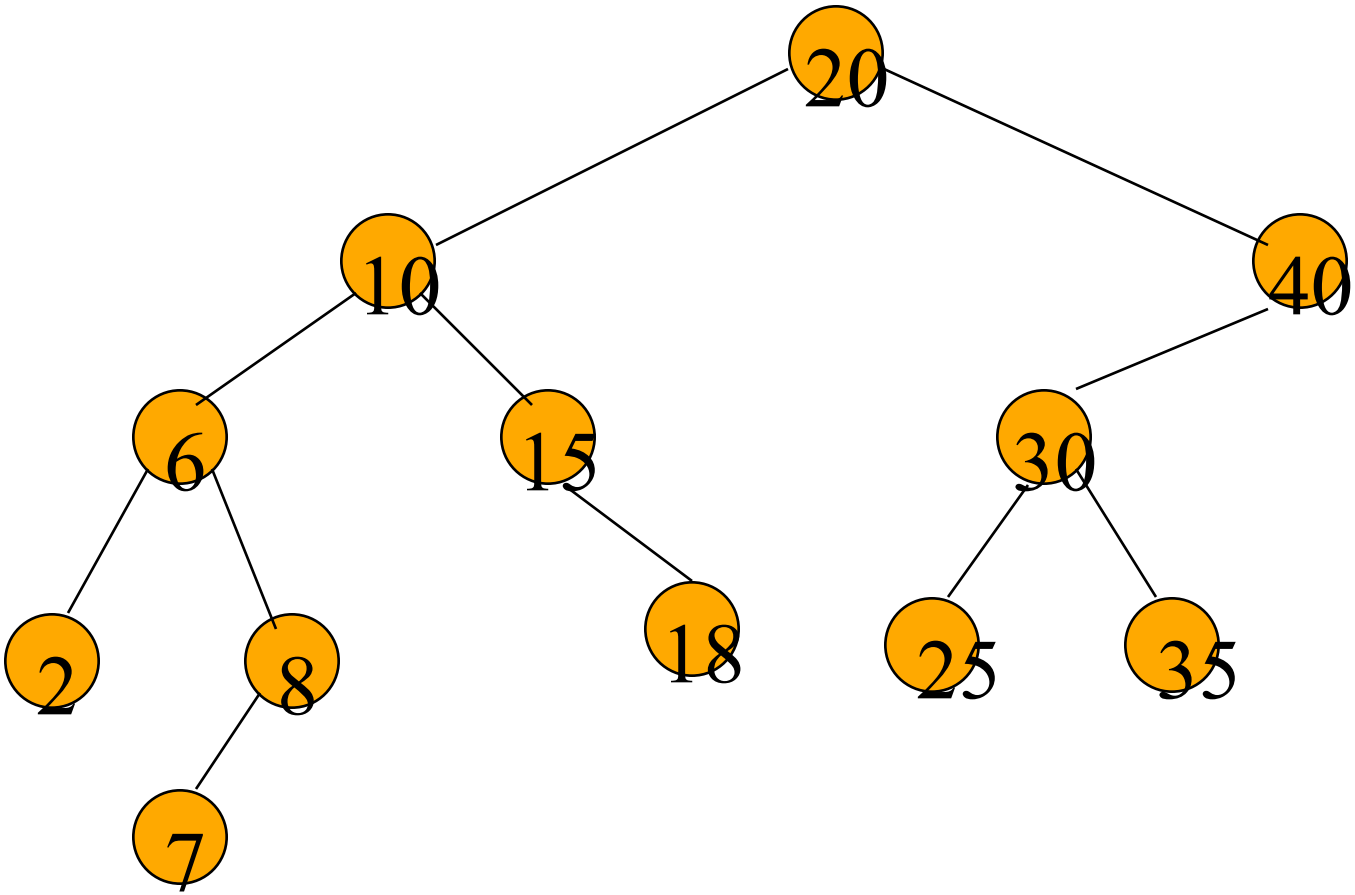
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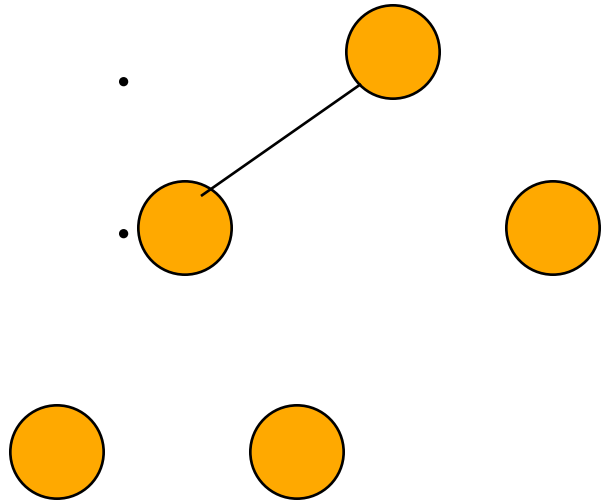

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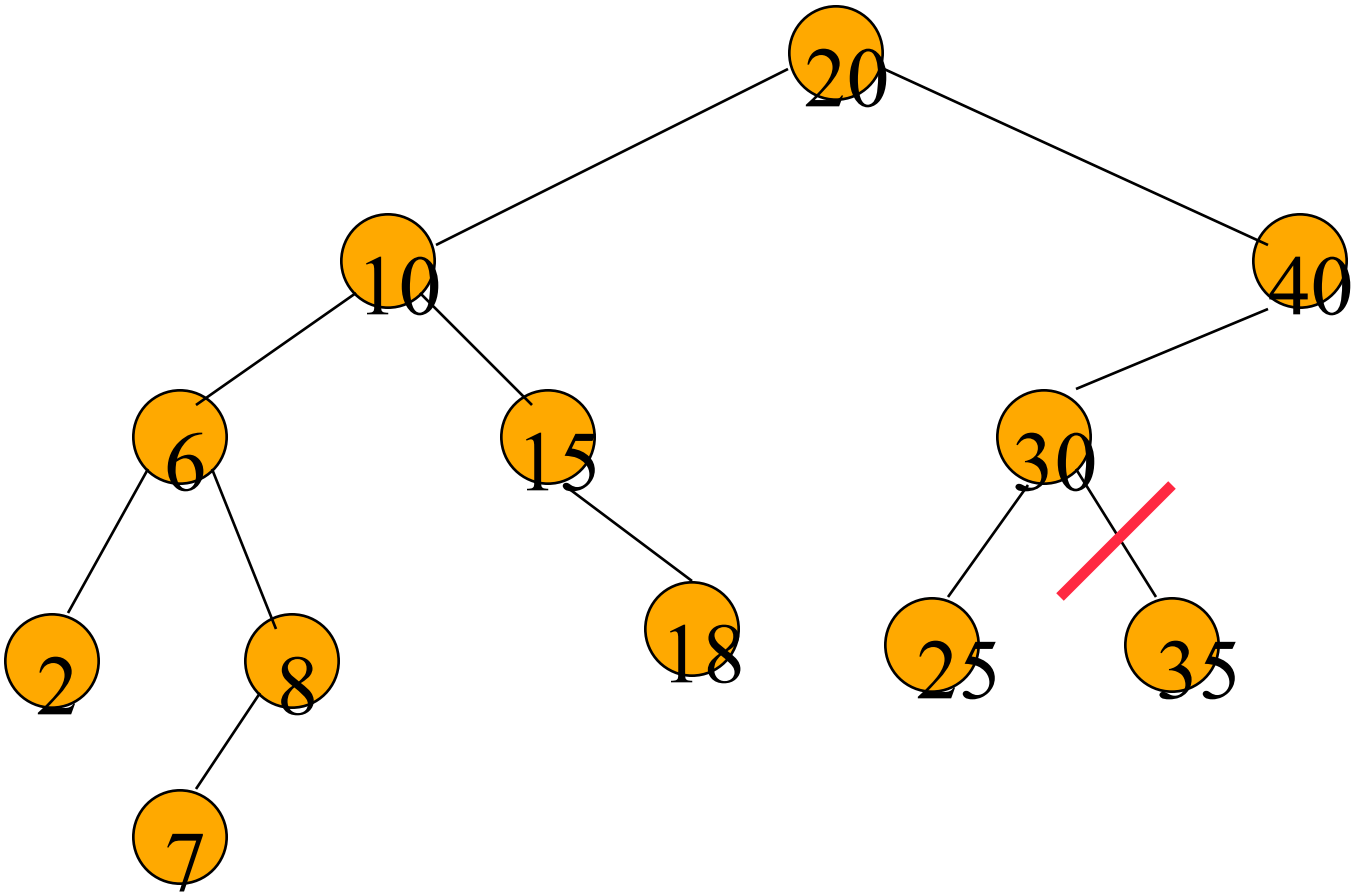

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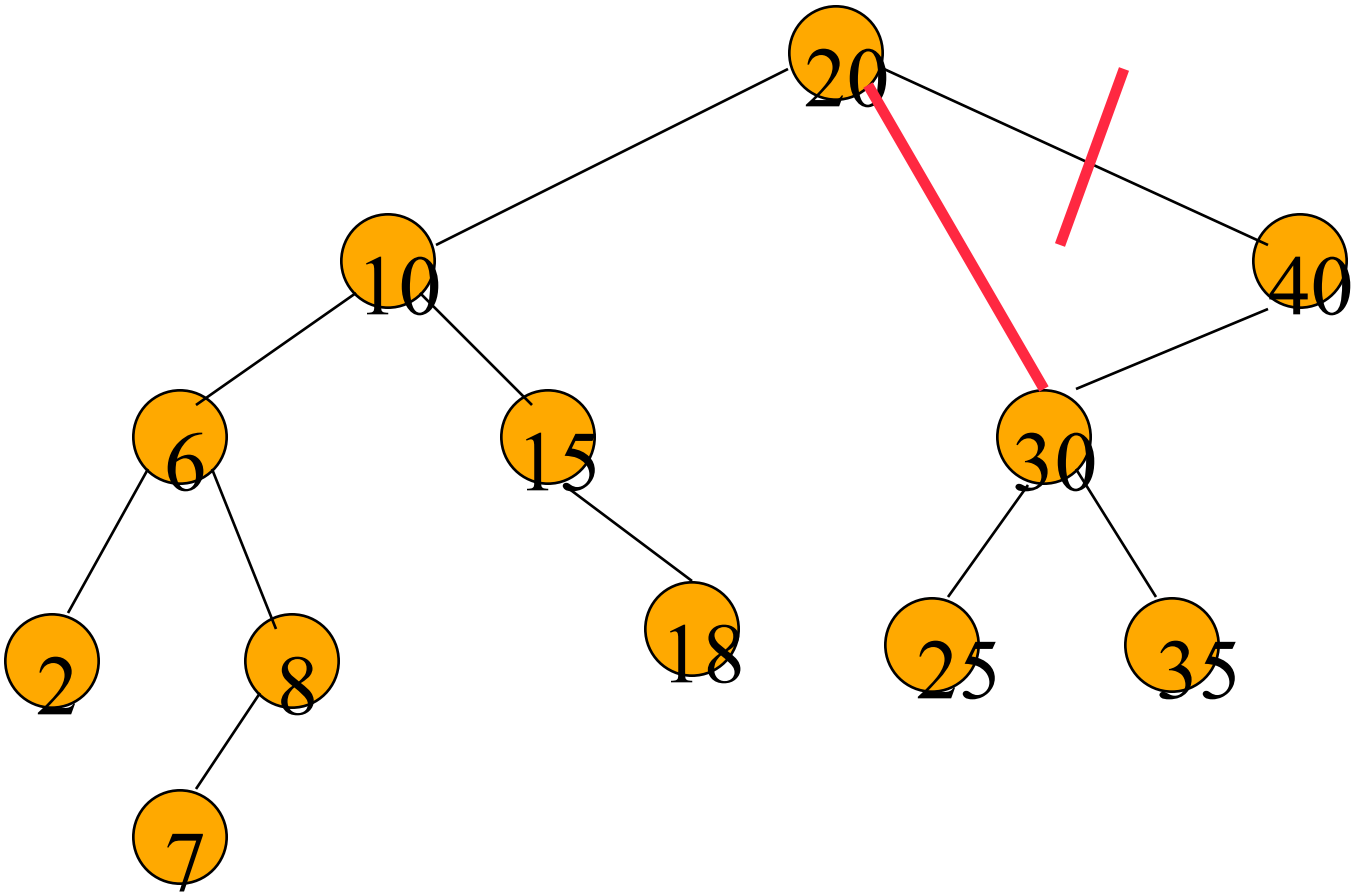

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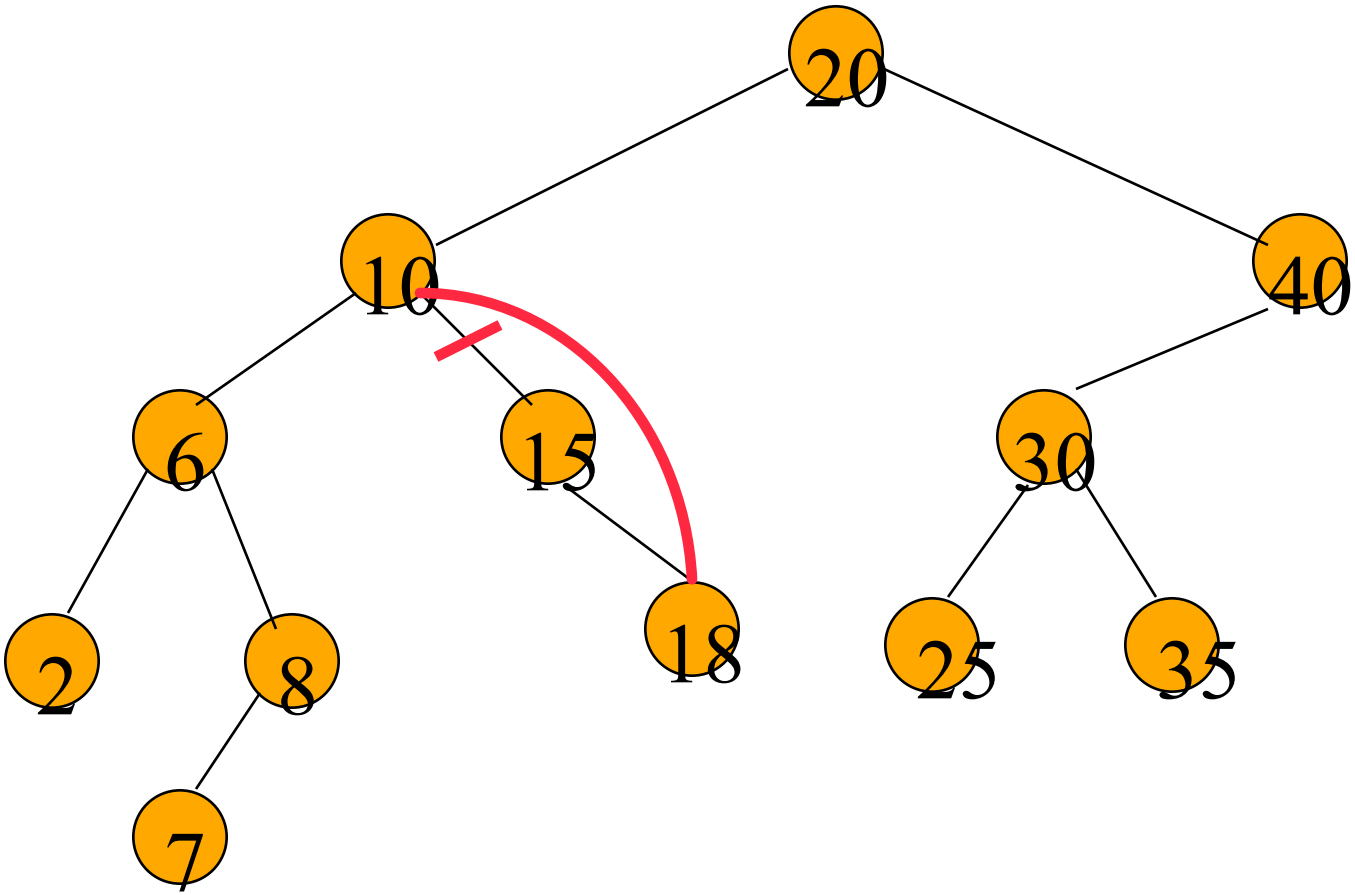

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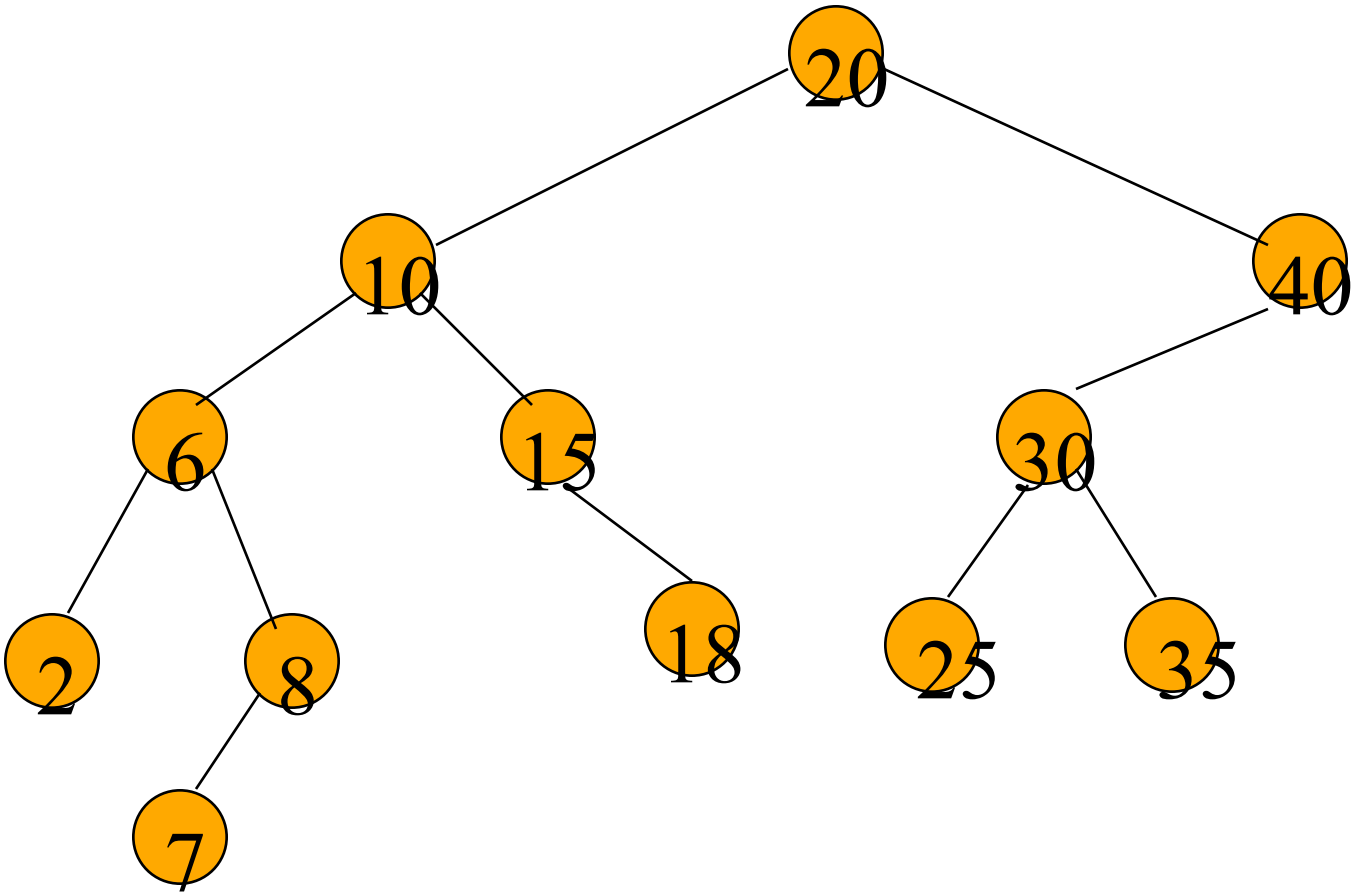

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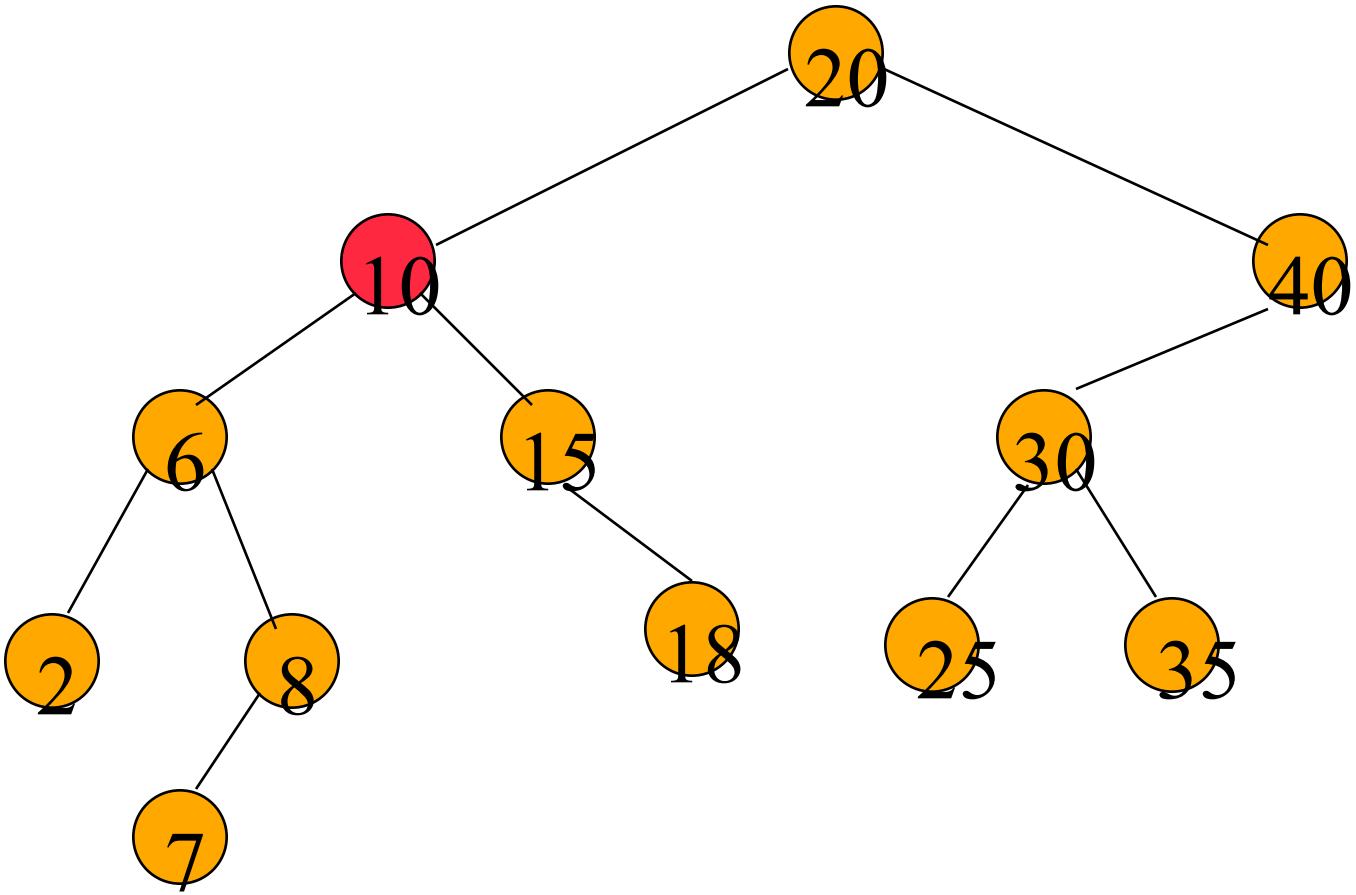

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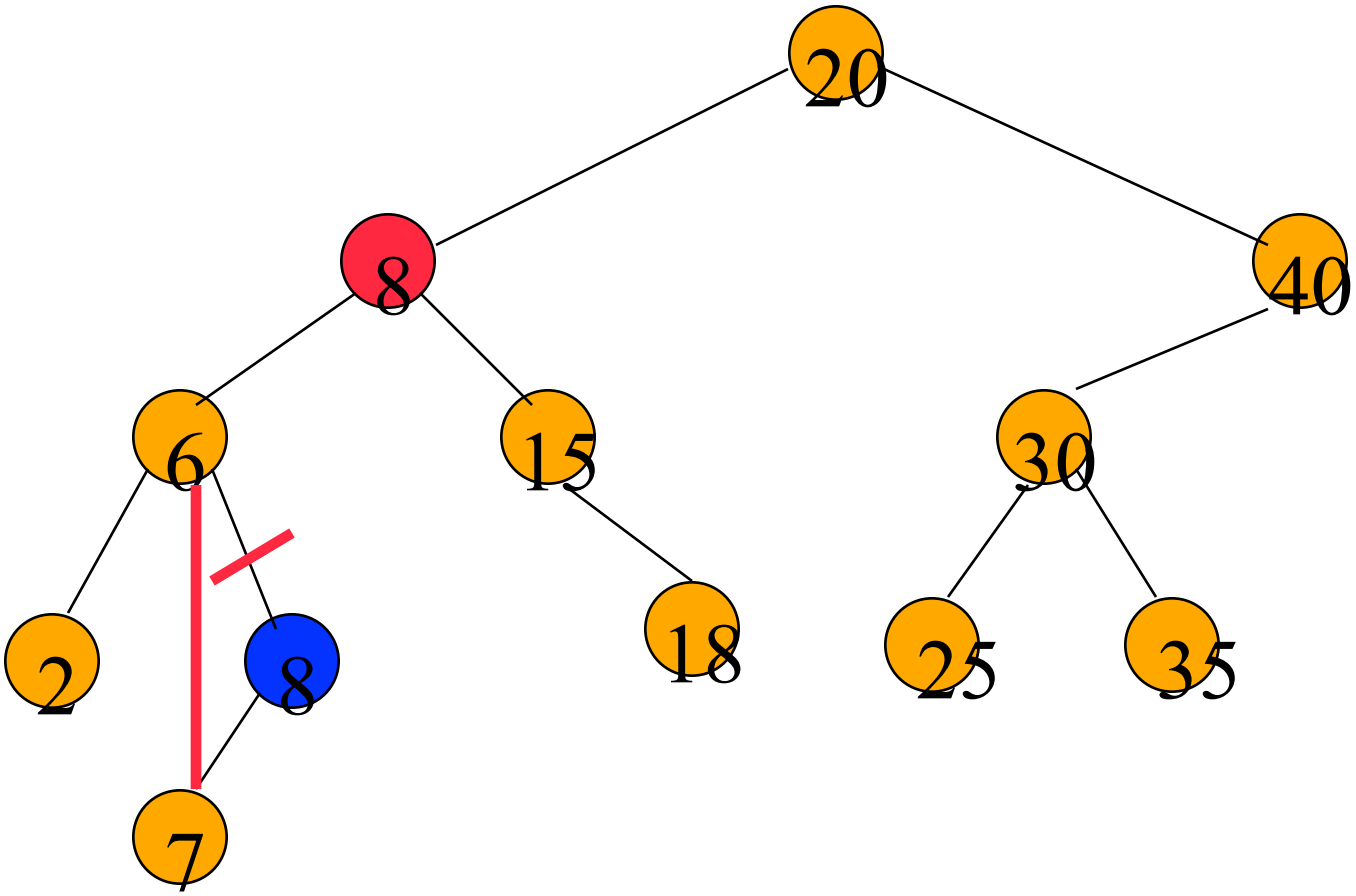

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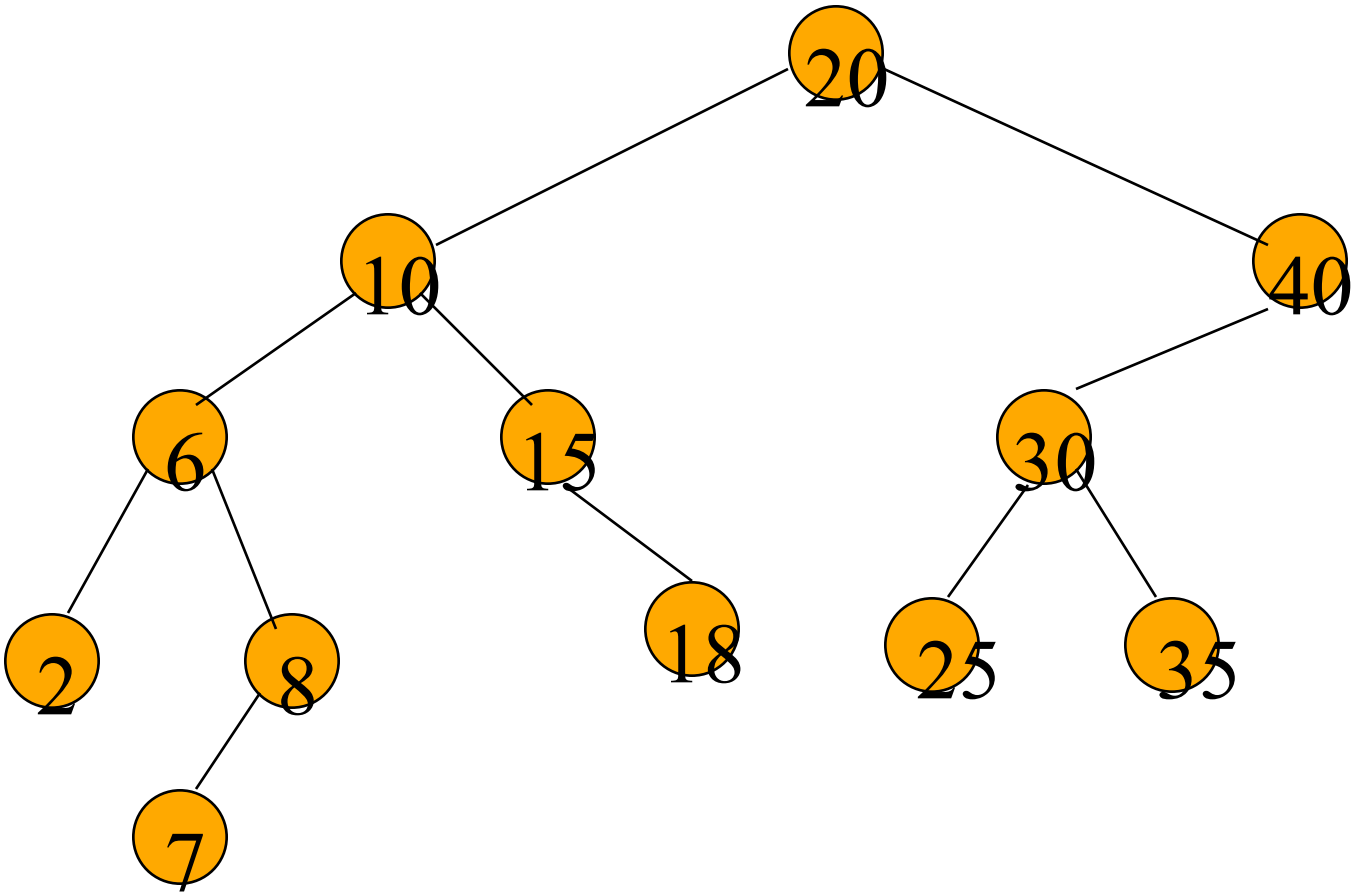

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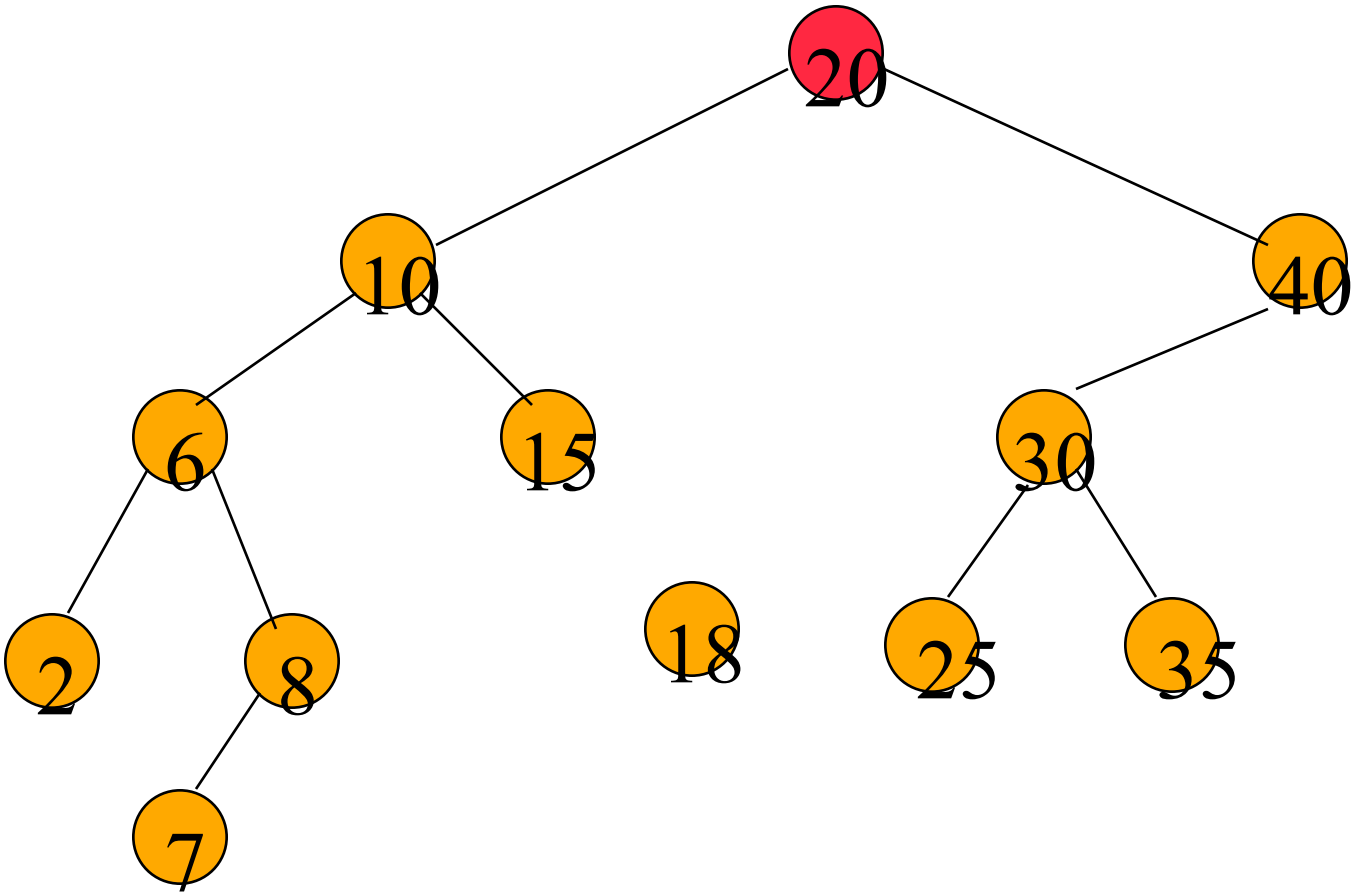
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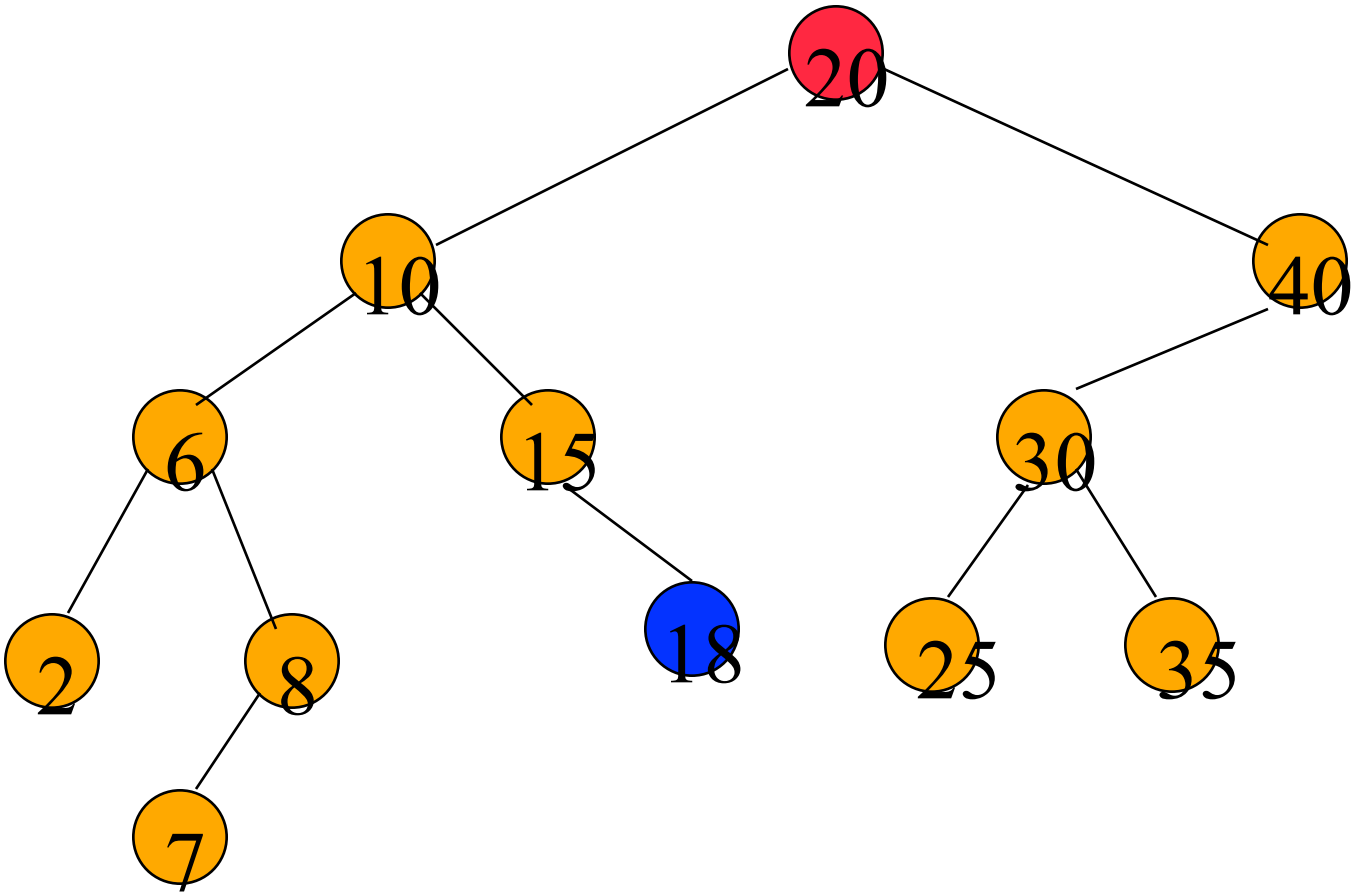
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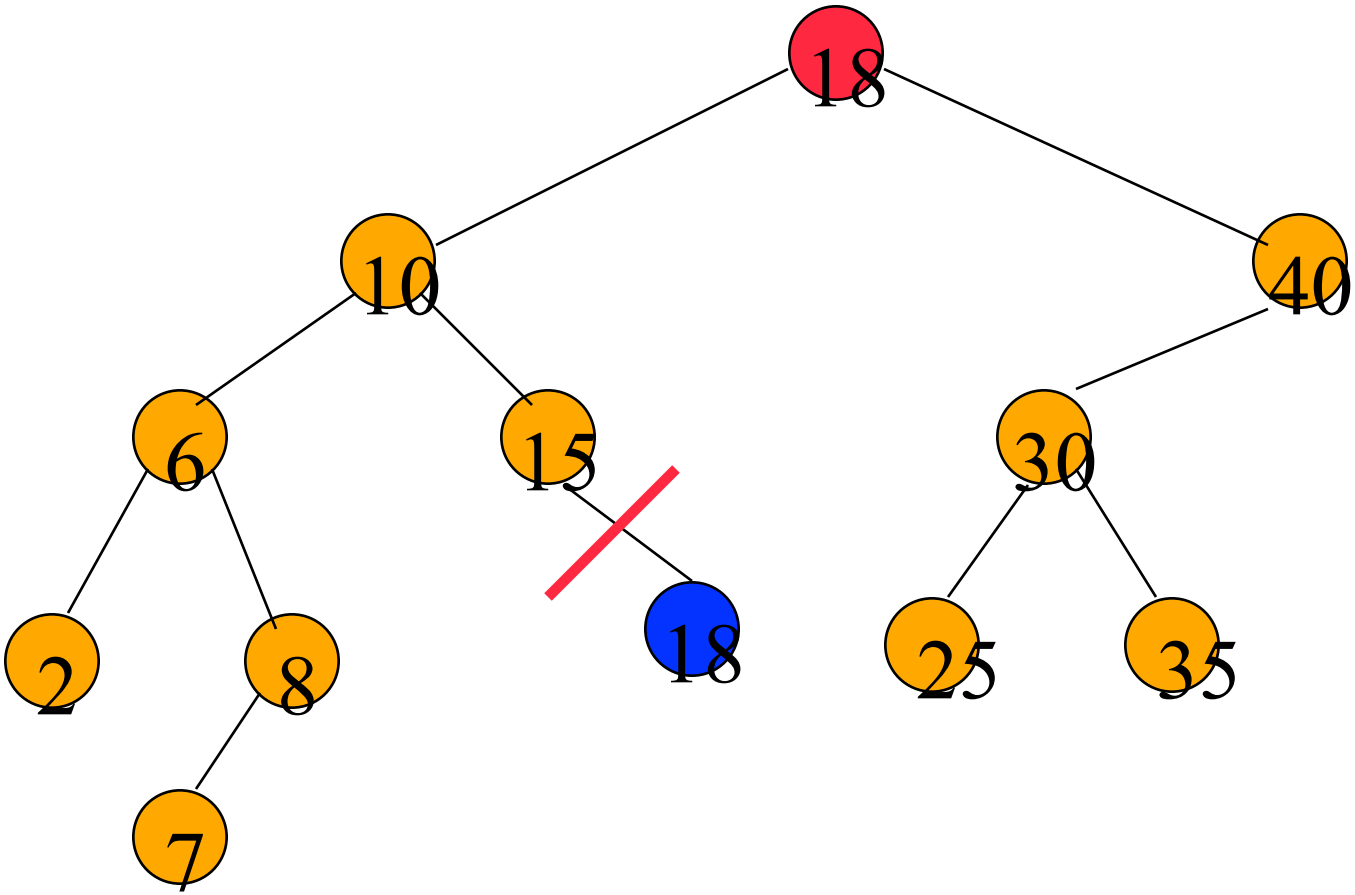
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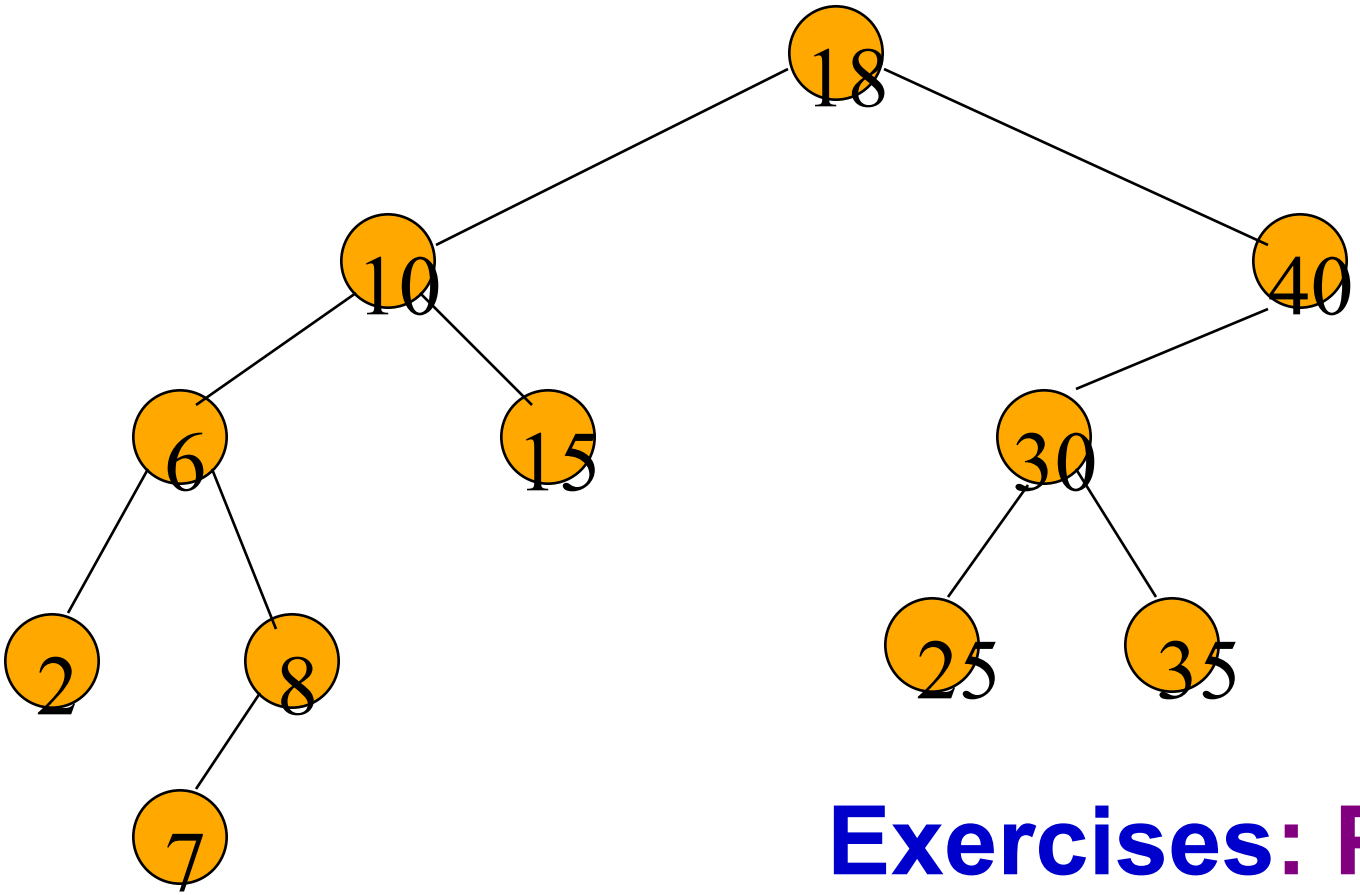

Exercises: P296-1,2

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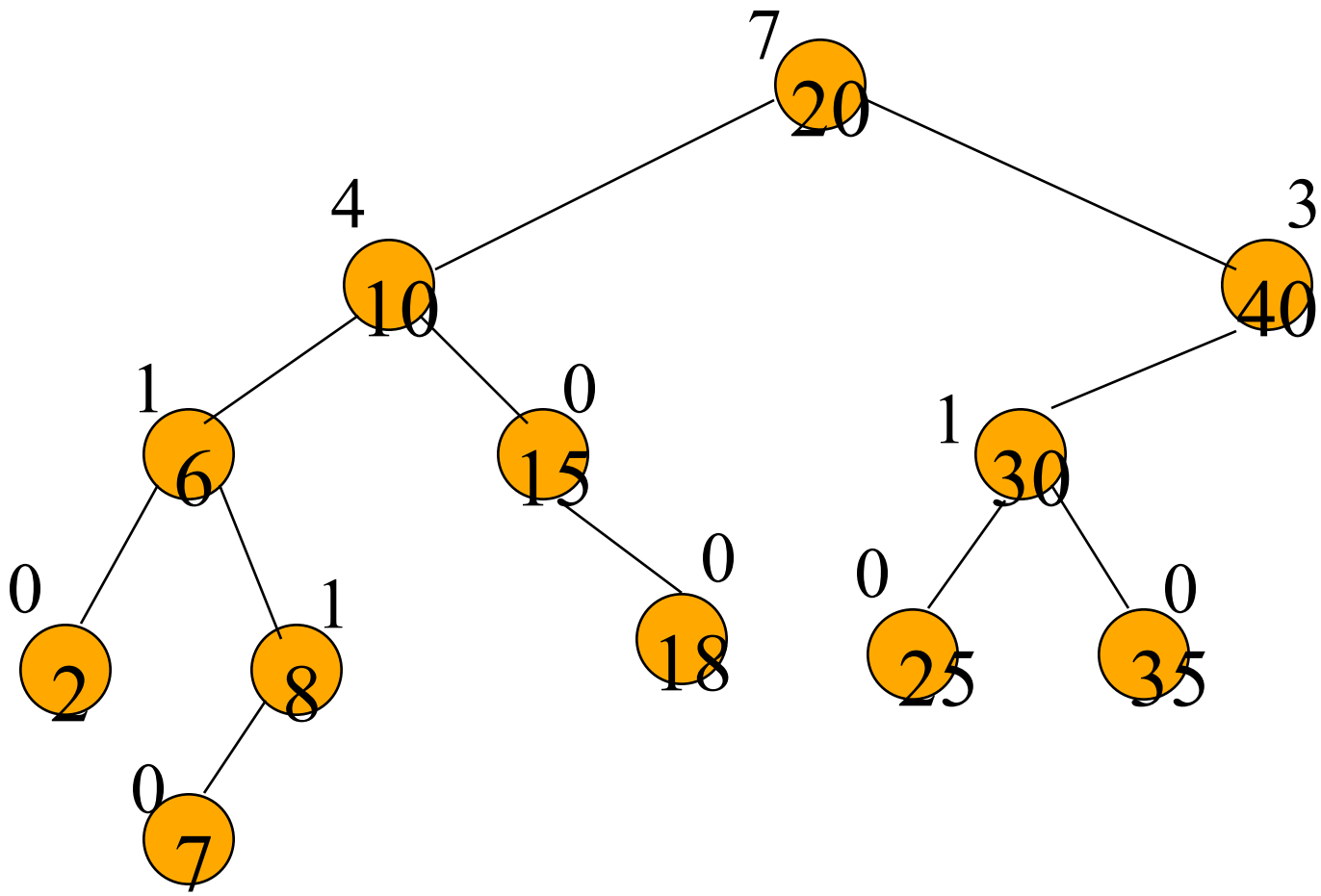
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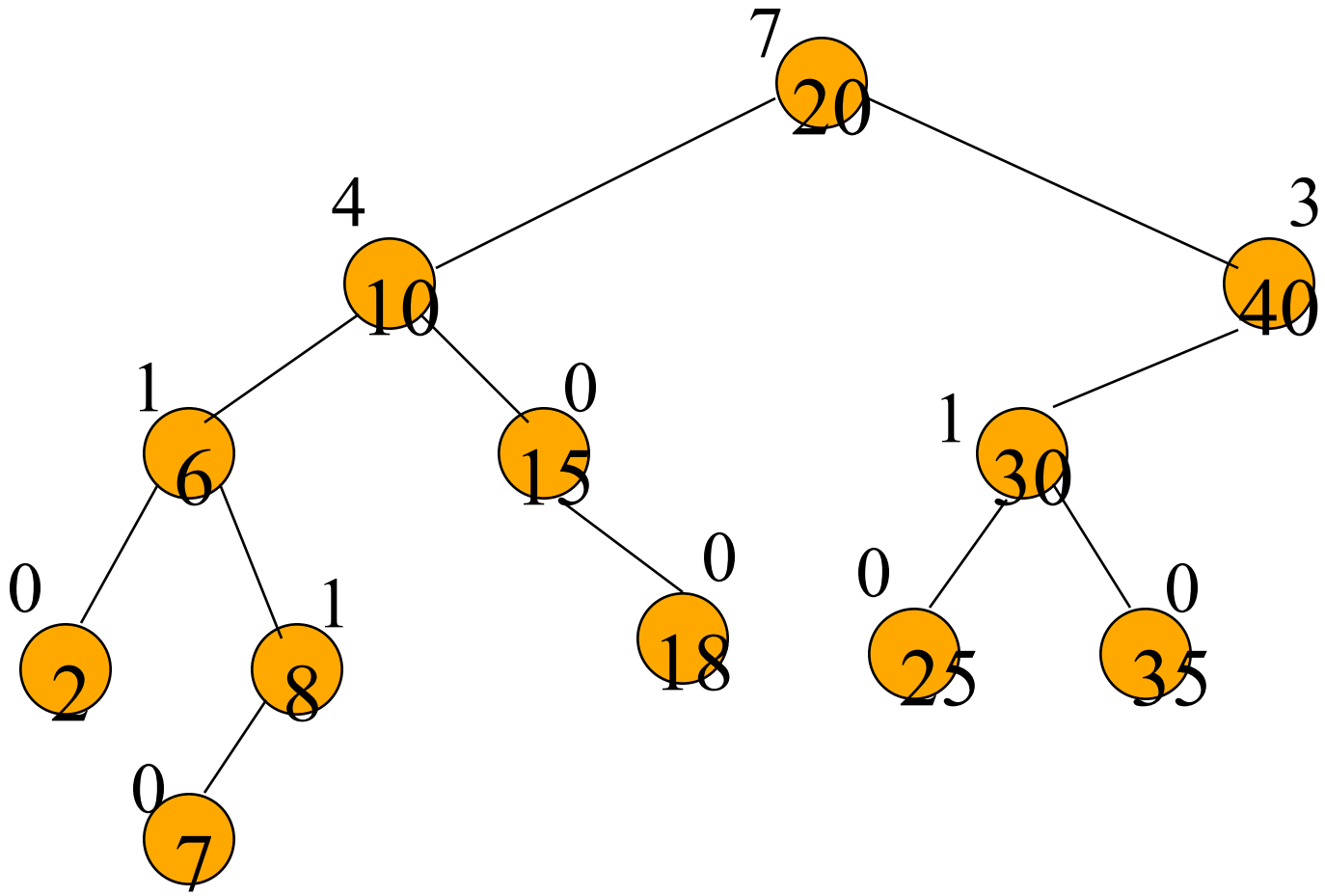

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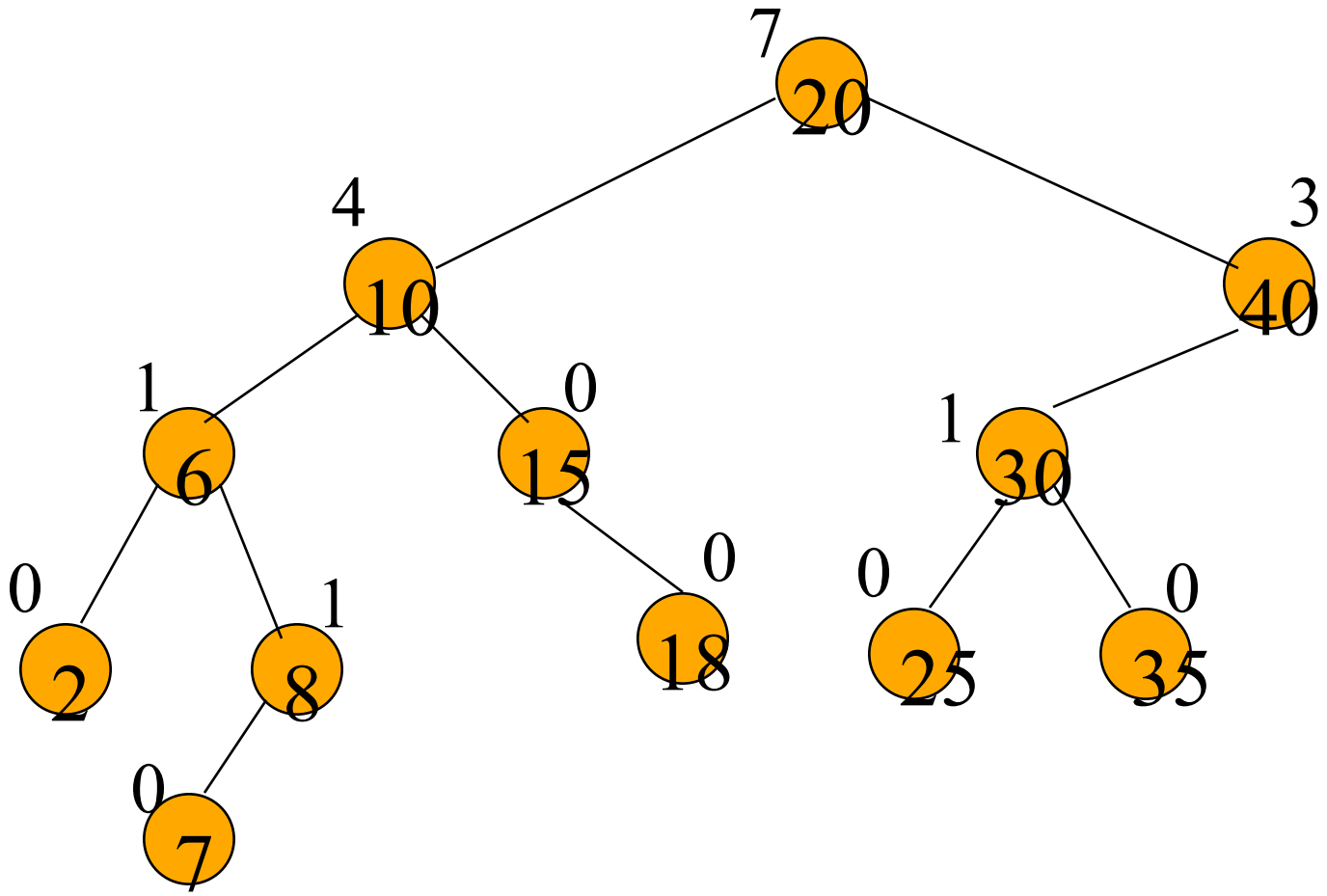

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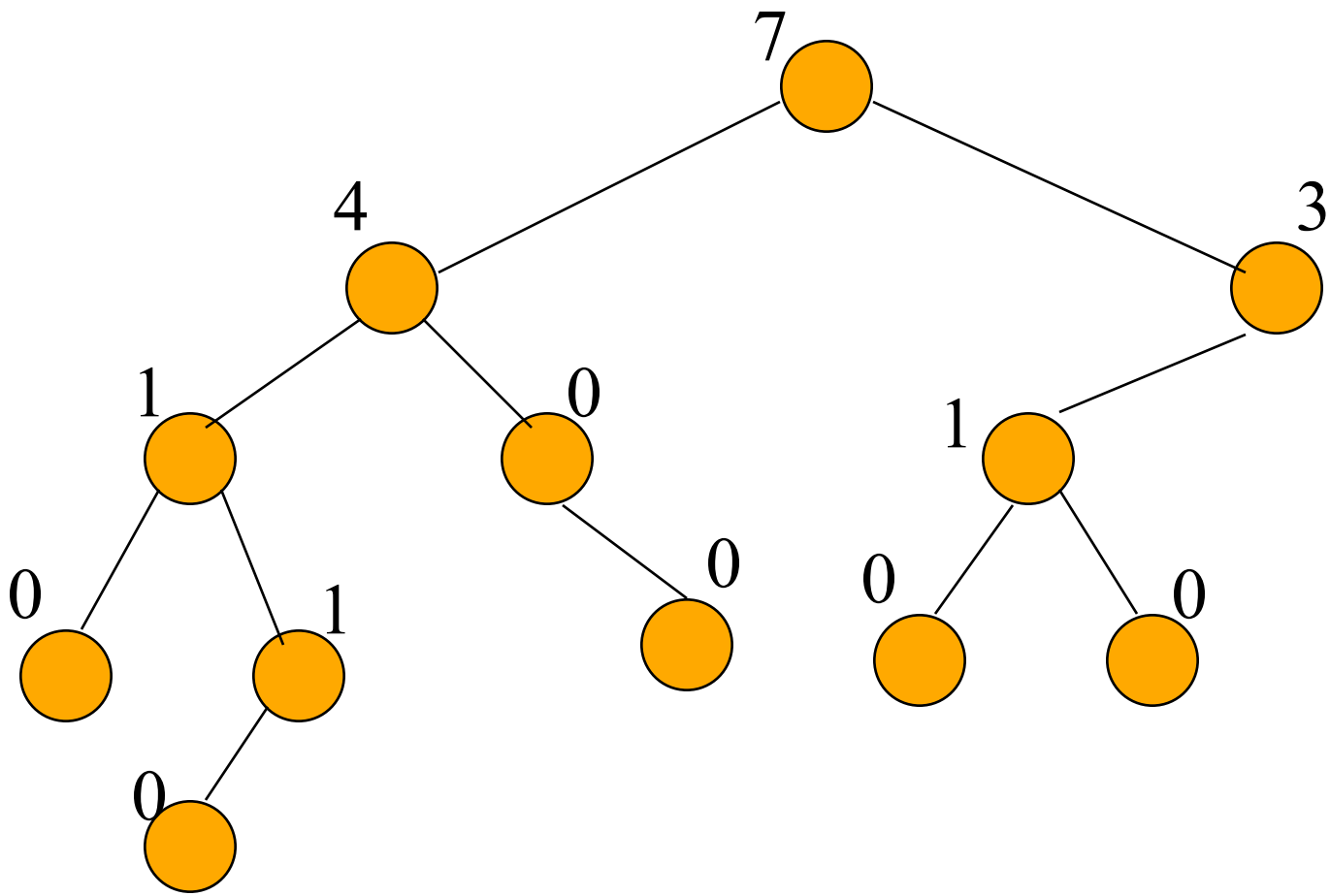
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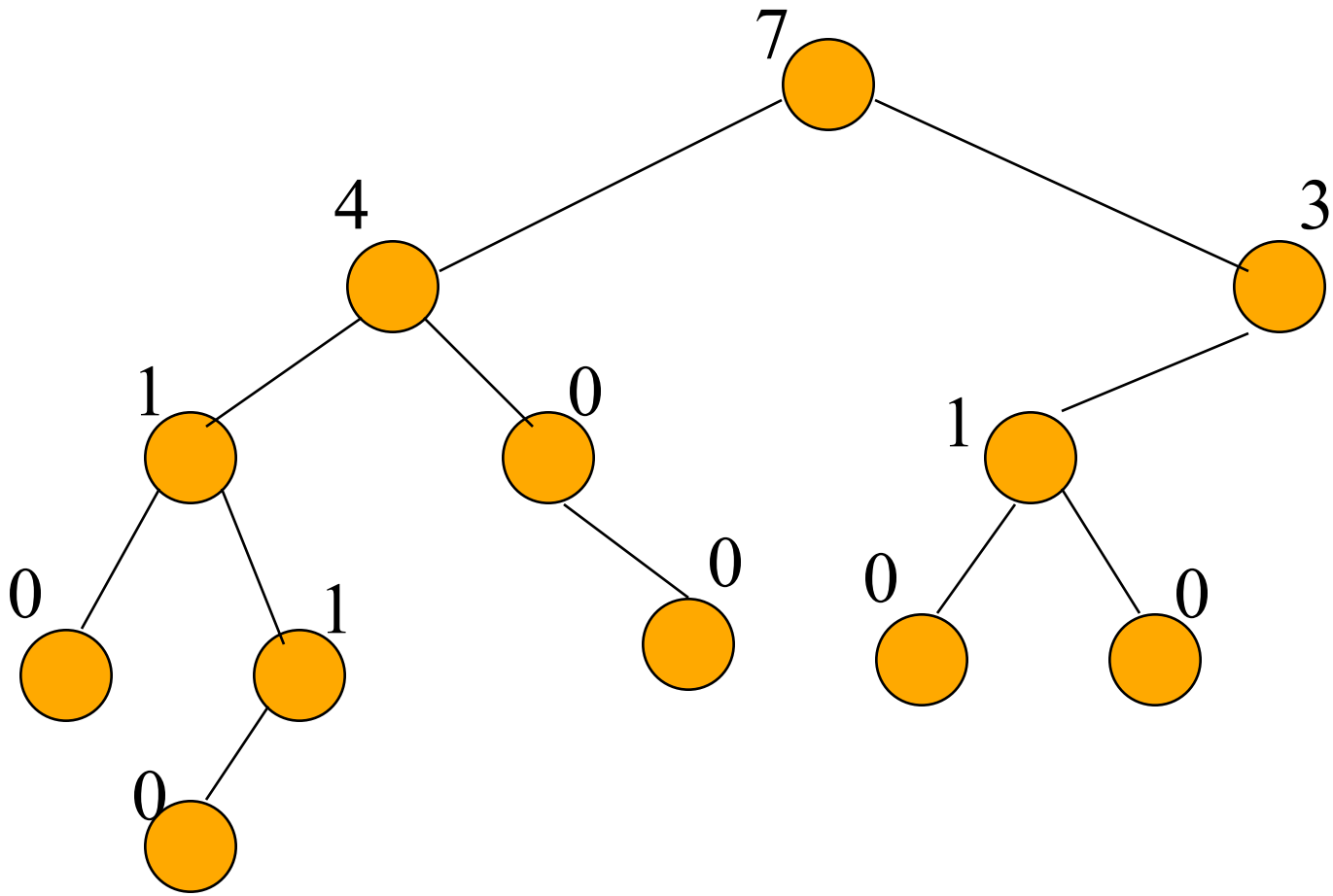

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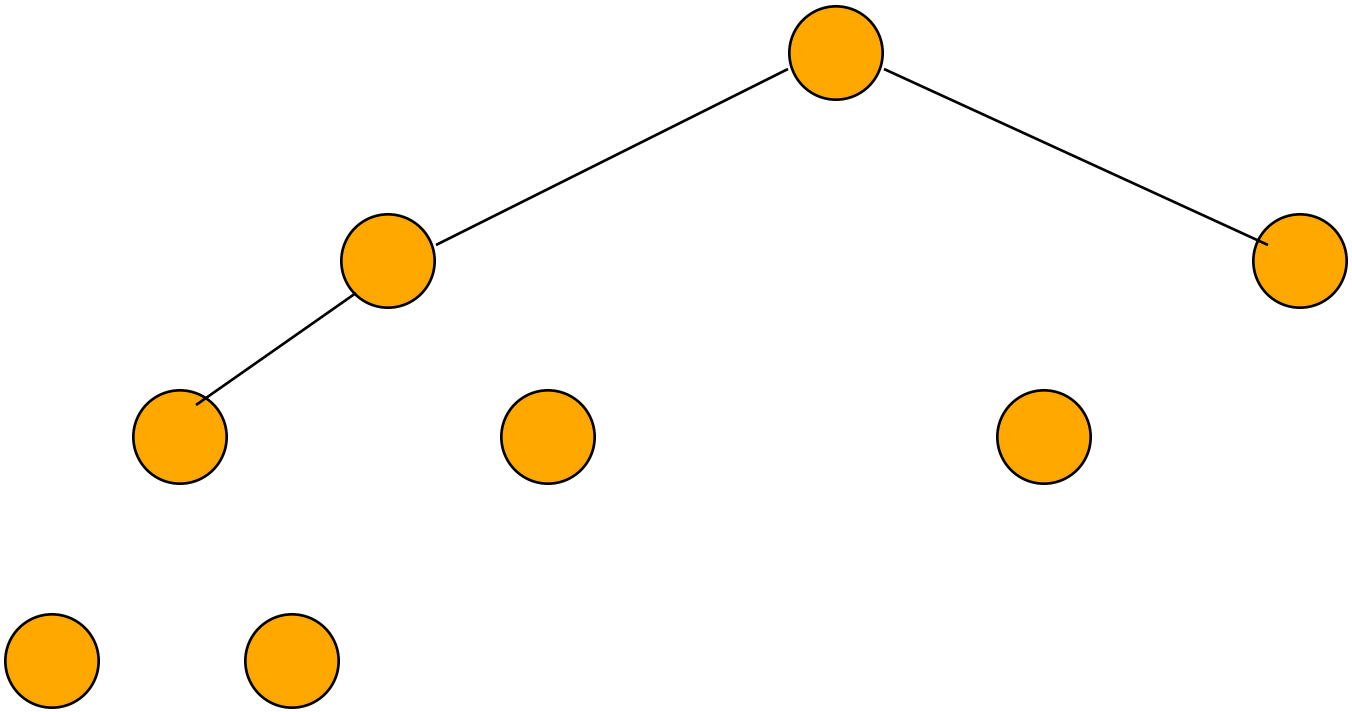

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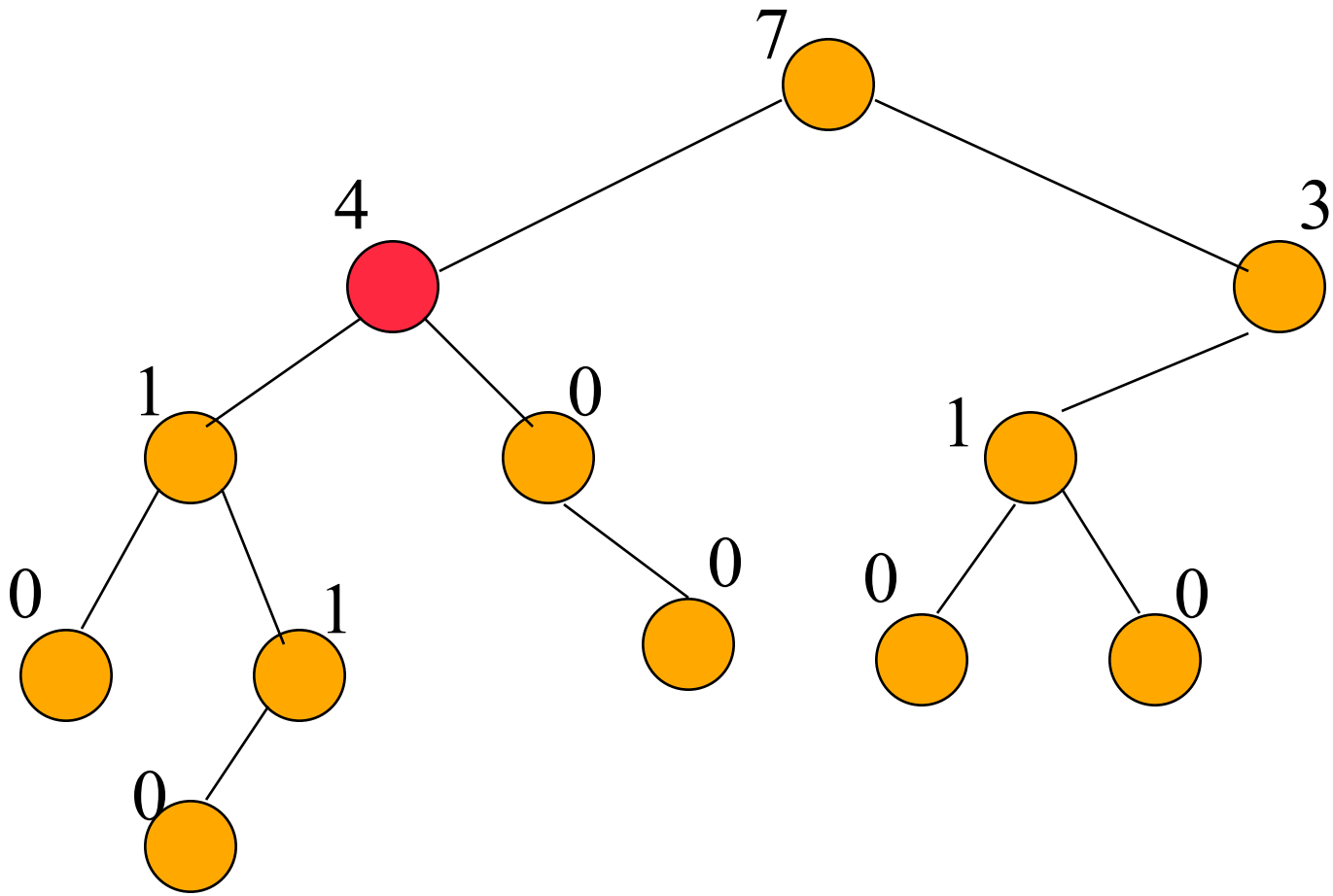

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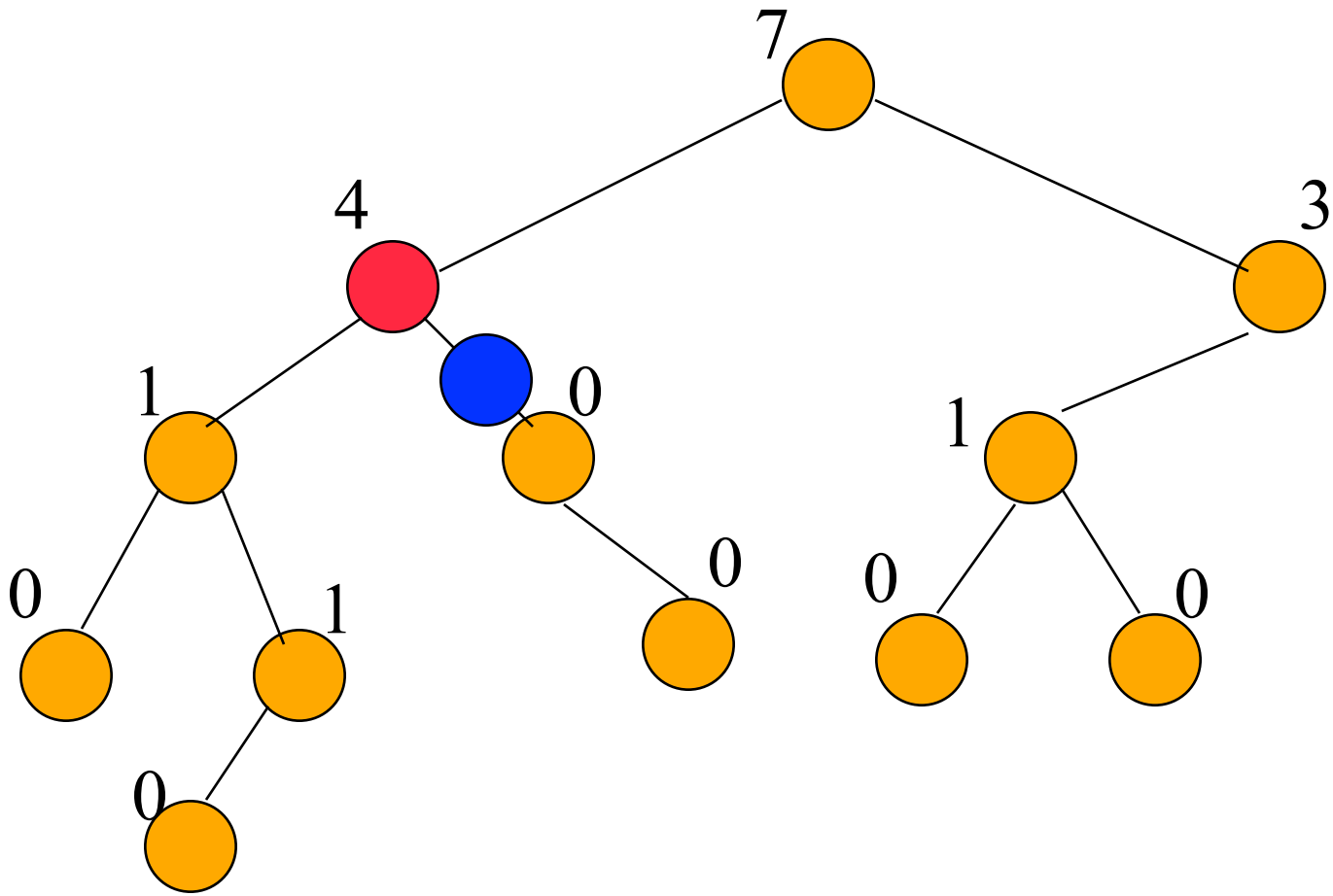

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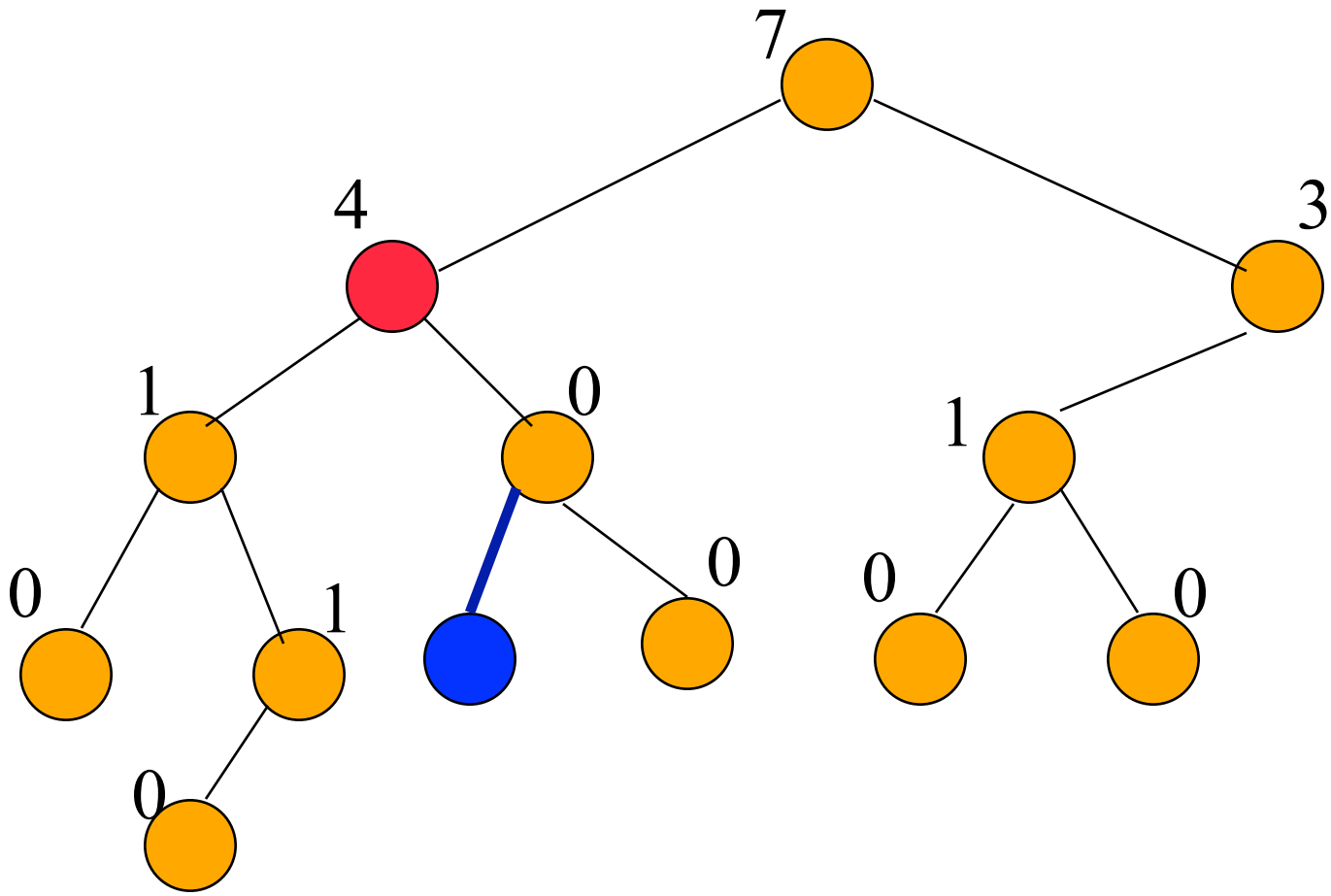

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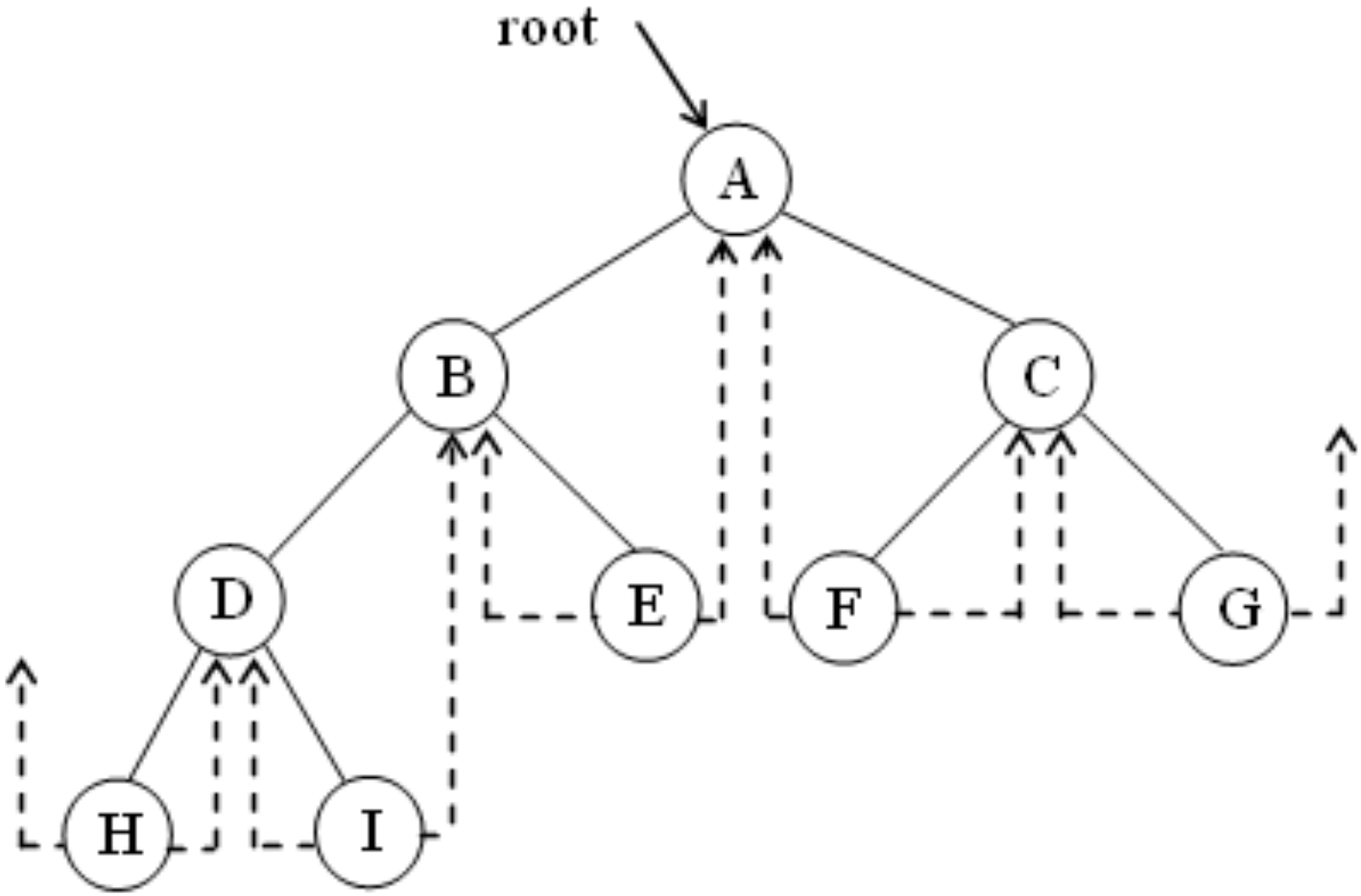
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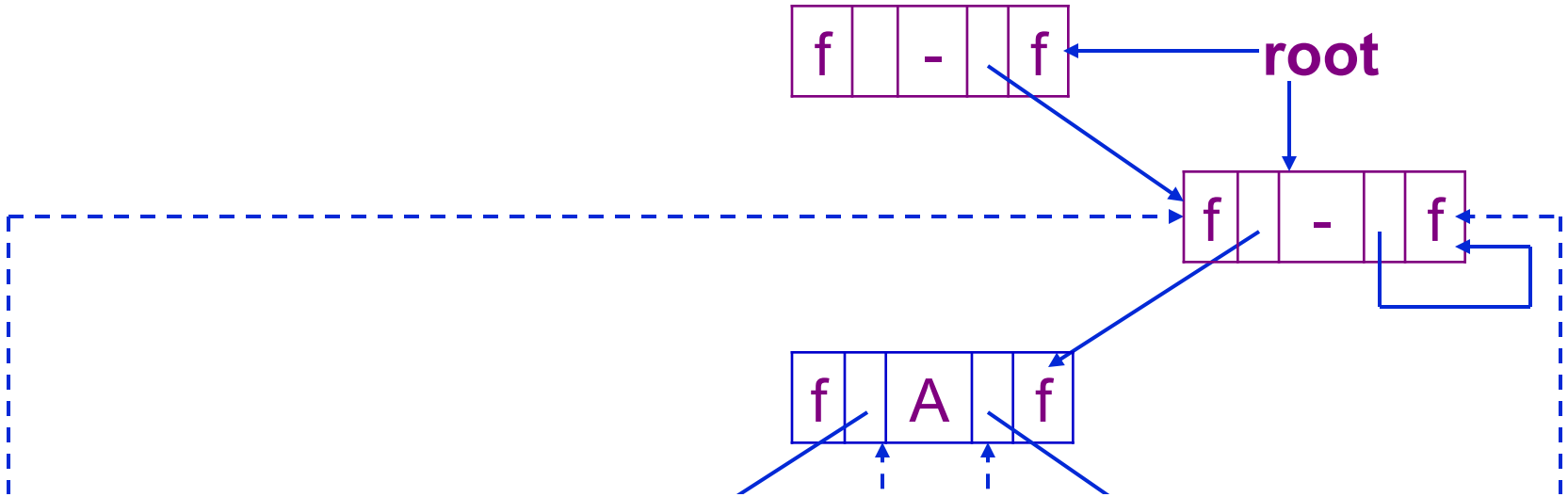
Let ThreadedInorderIterator be a nested class of ThreadedTree:

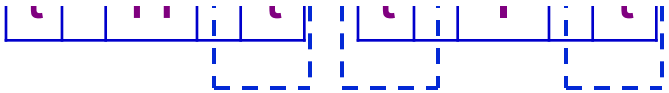
An empty threaded binary tree



we can see:

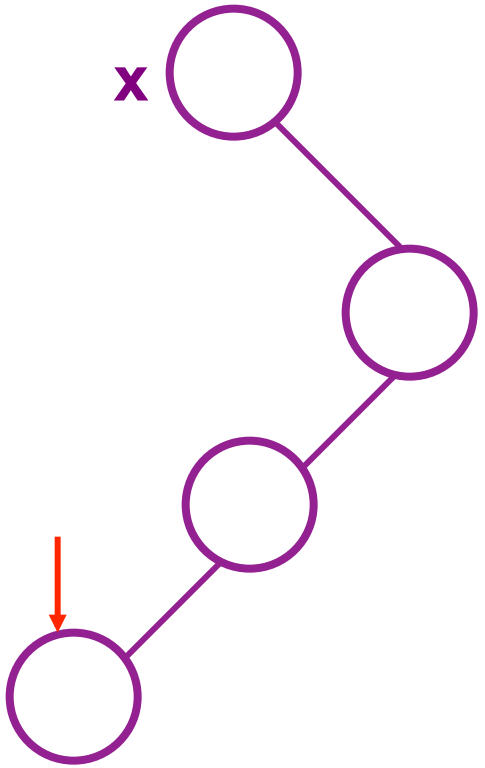
(1)The inorder successor of the head node is the first node in inorder;

(2)The inorder successor of the last node in inorder is the head node.



memory representation of threaded tree

Inorder Traversal of a Threaded Binary Tree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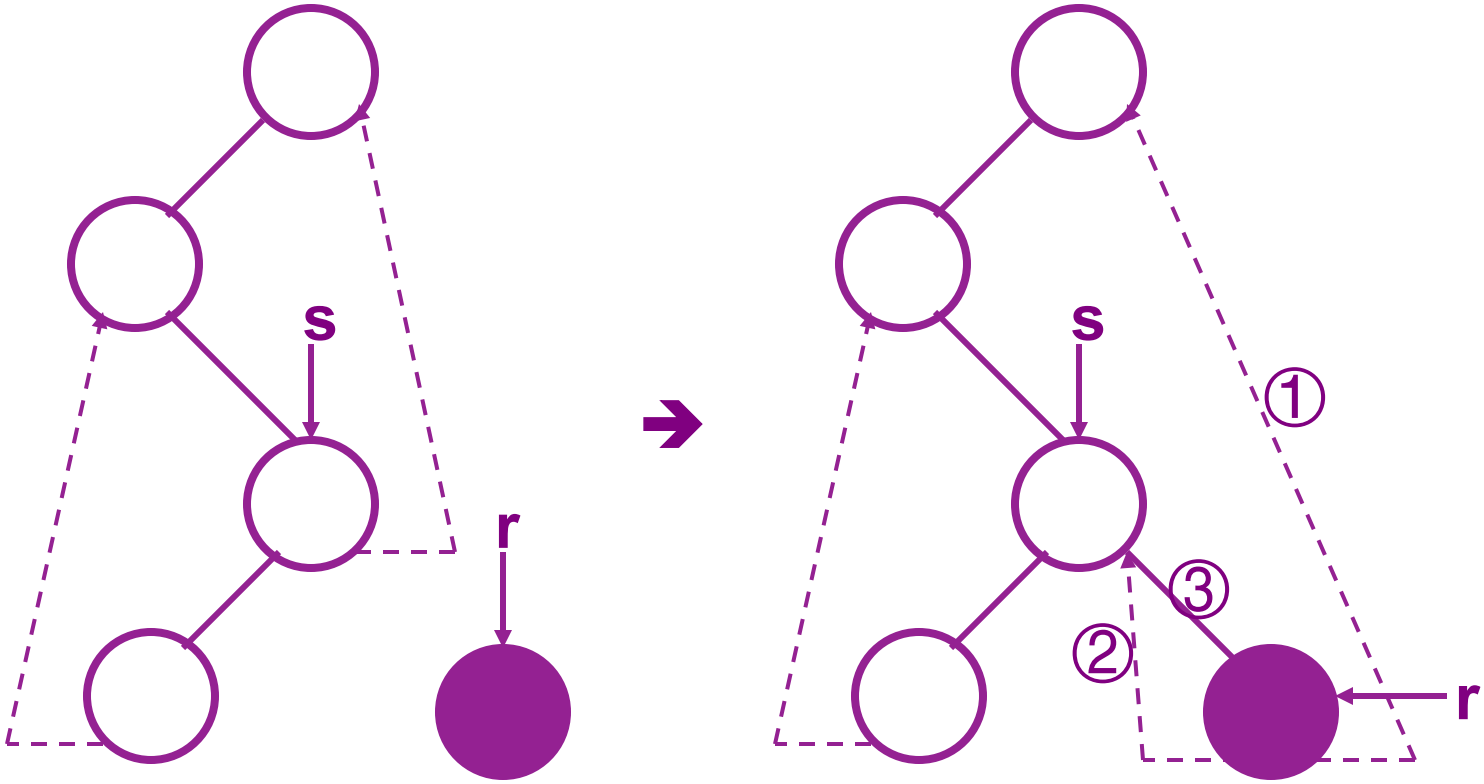
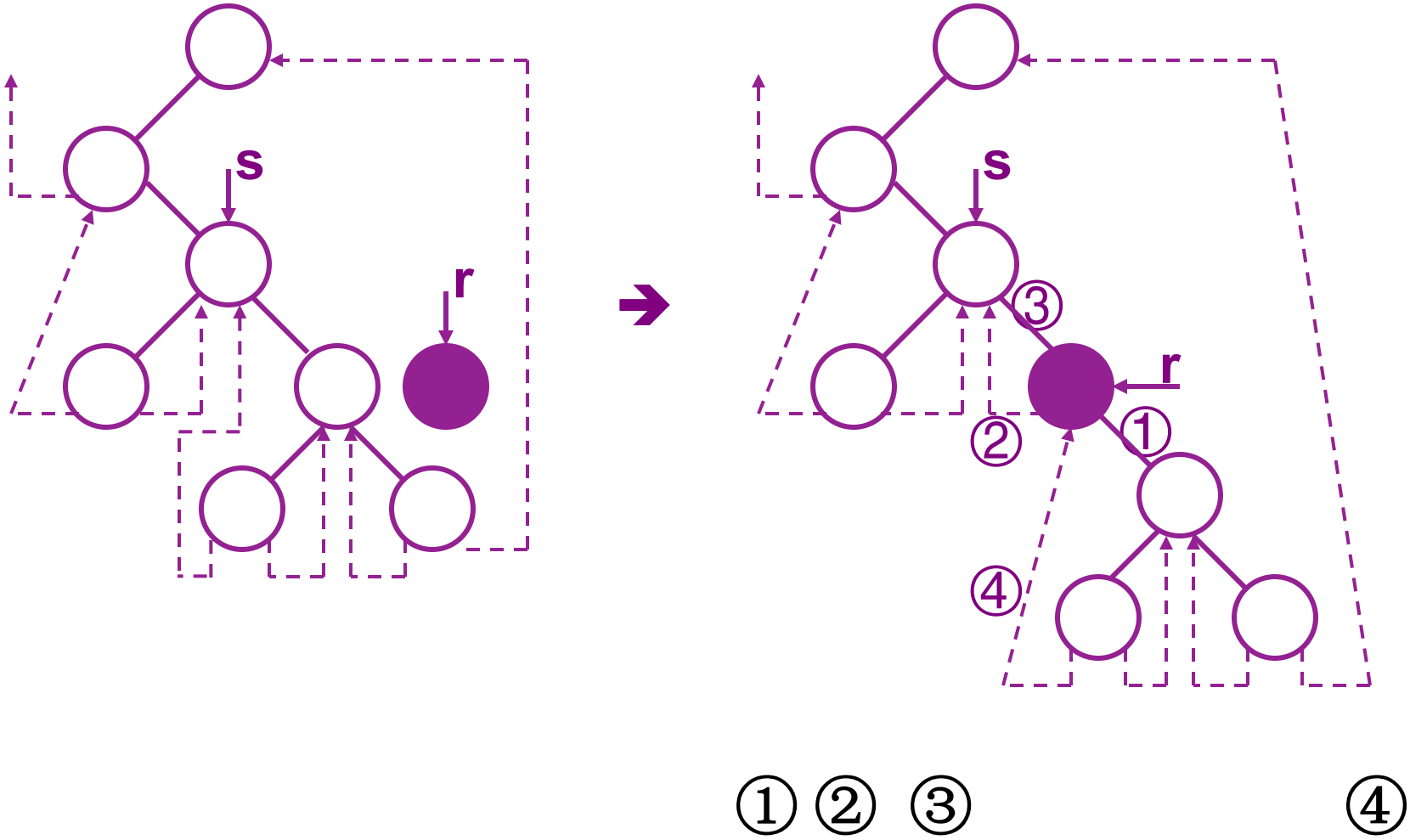
Inserting a Node into a Threaded Binary Tree

(1) If s rightThread==true, as:



(2) If s　rightThread==false, as:



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Exercises: P277-1, P278-4

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ADT MaxHeap

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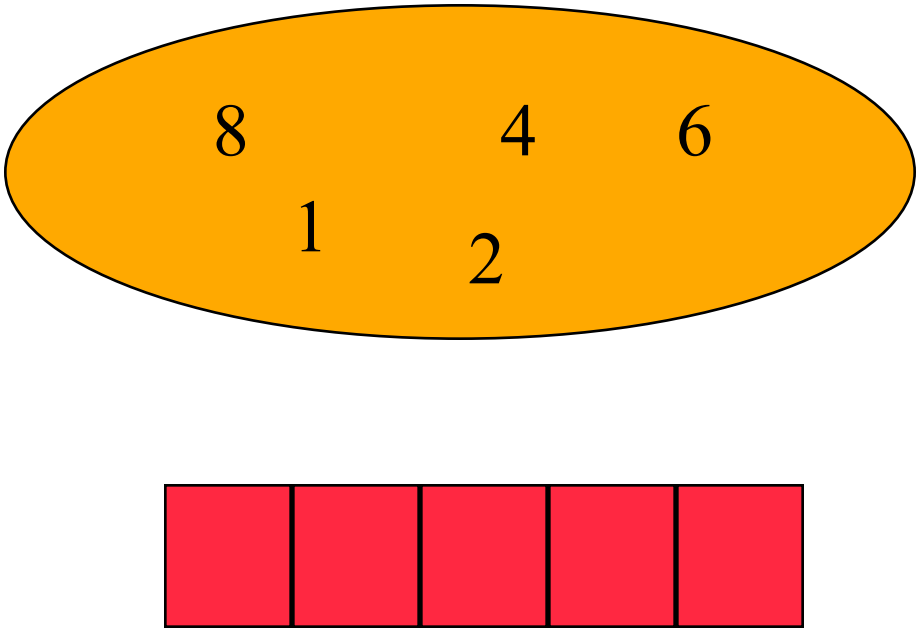
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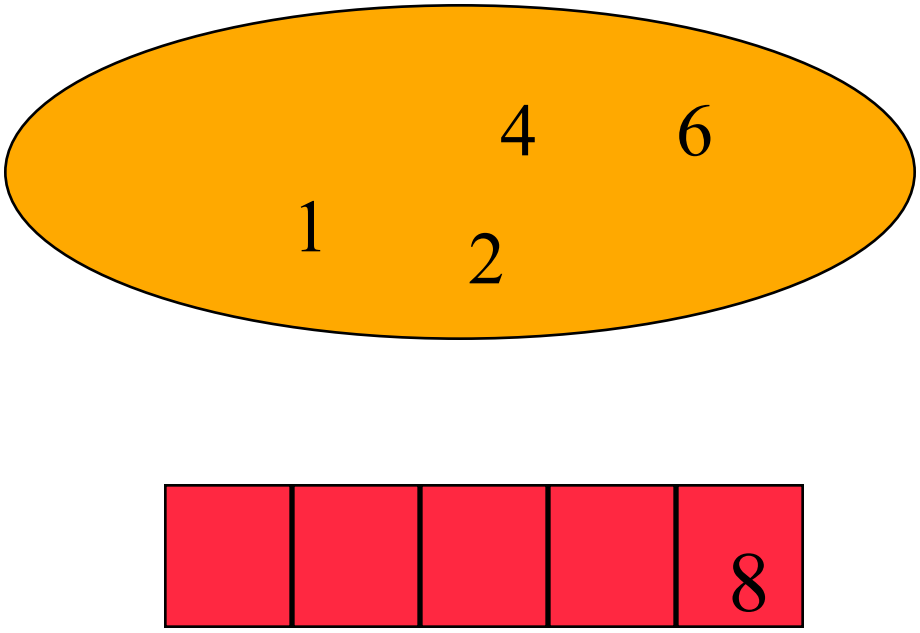
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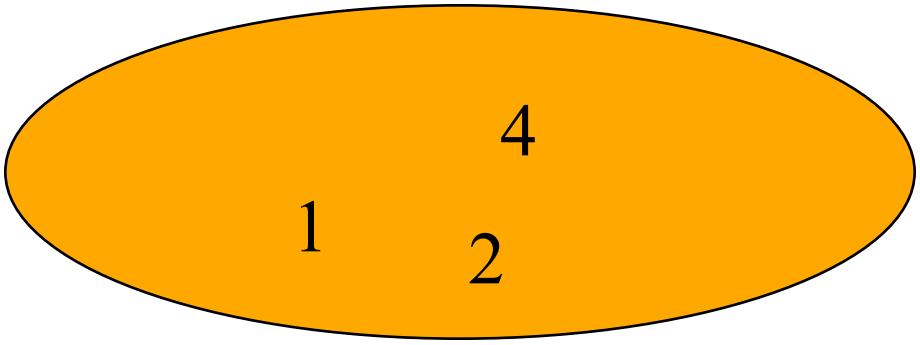
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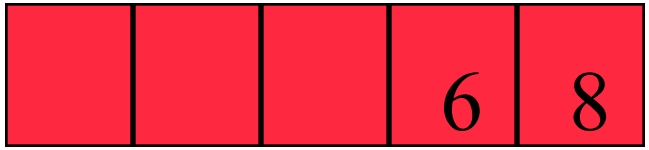
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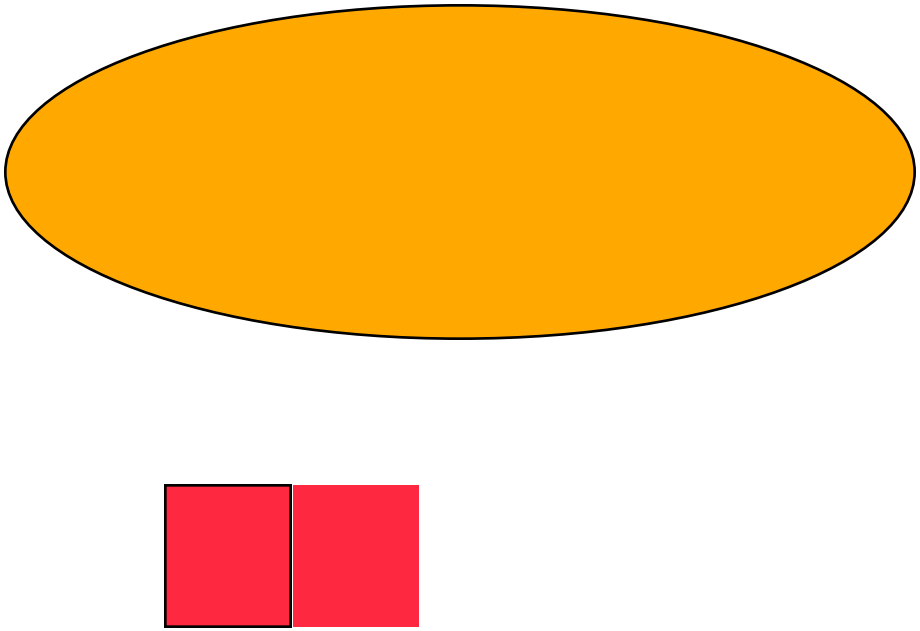

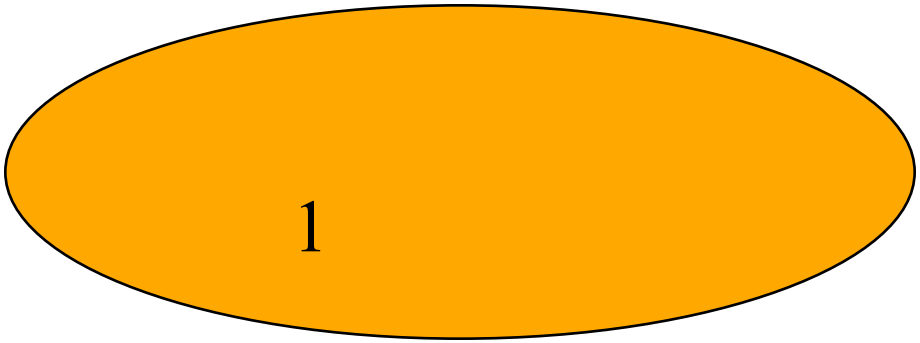
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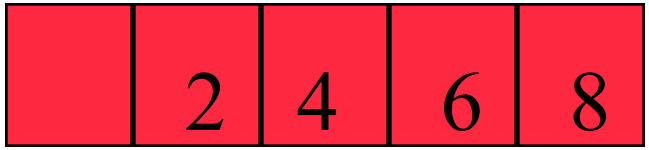
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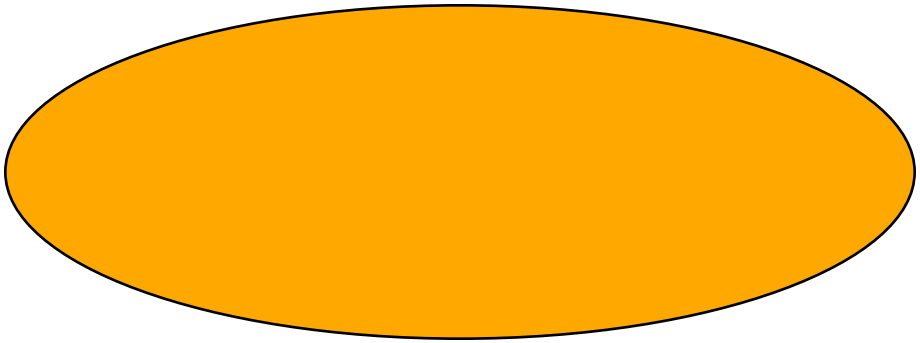

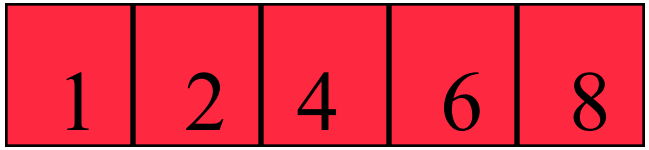
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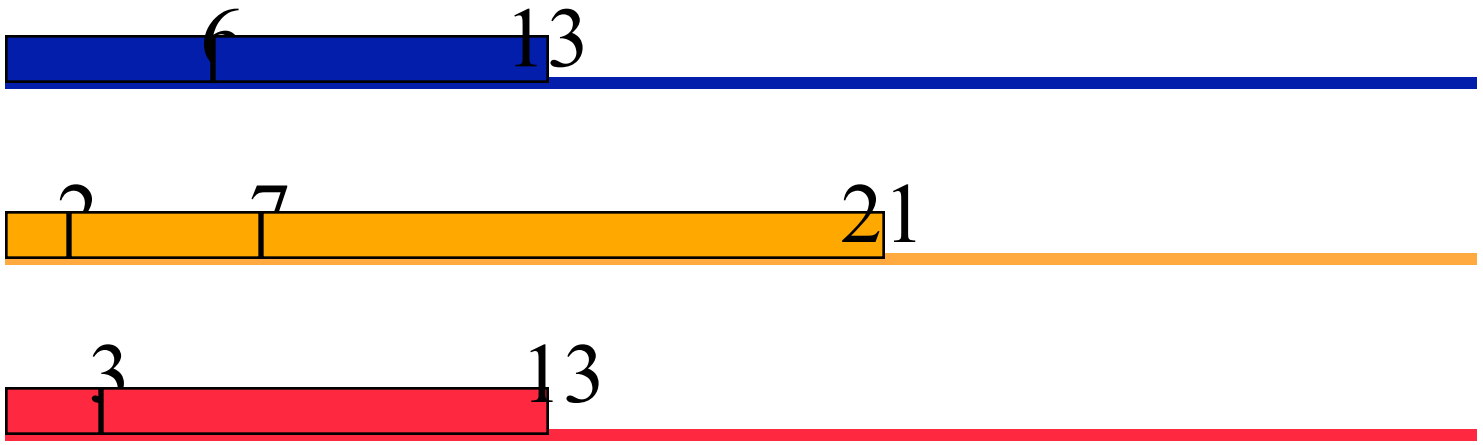

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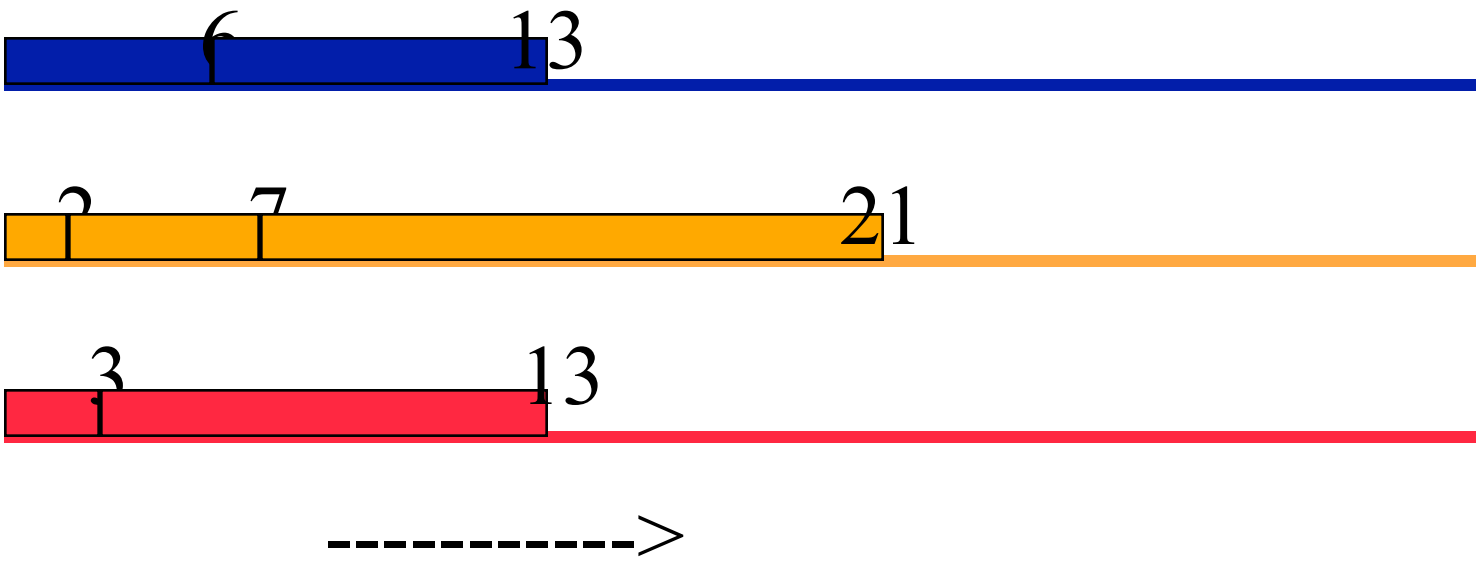
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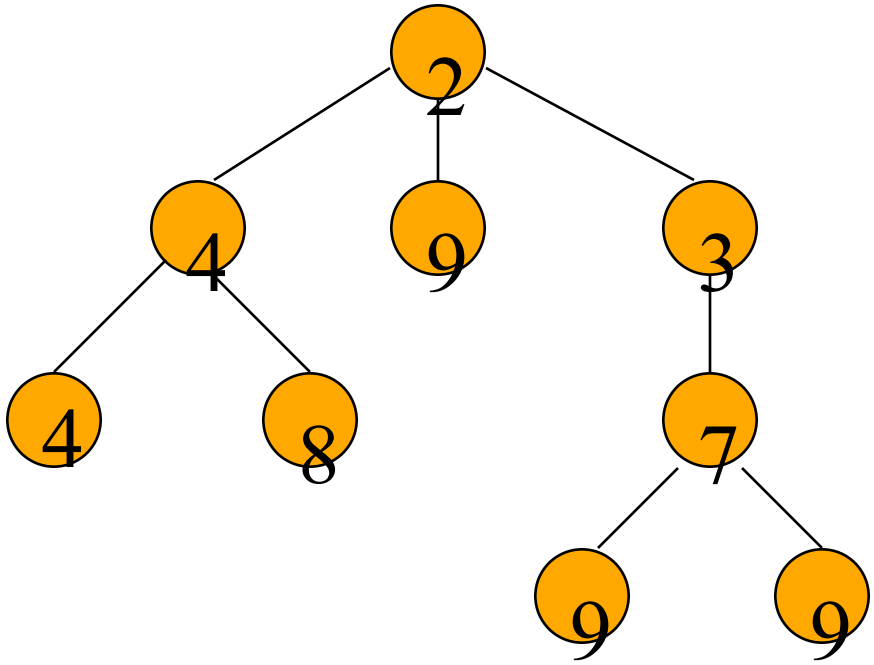
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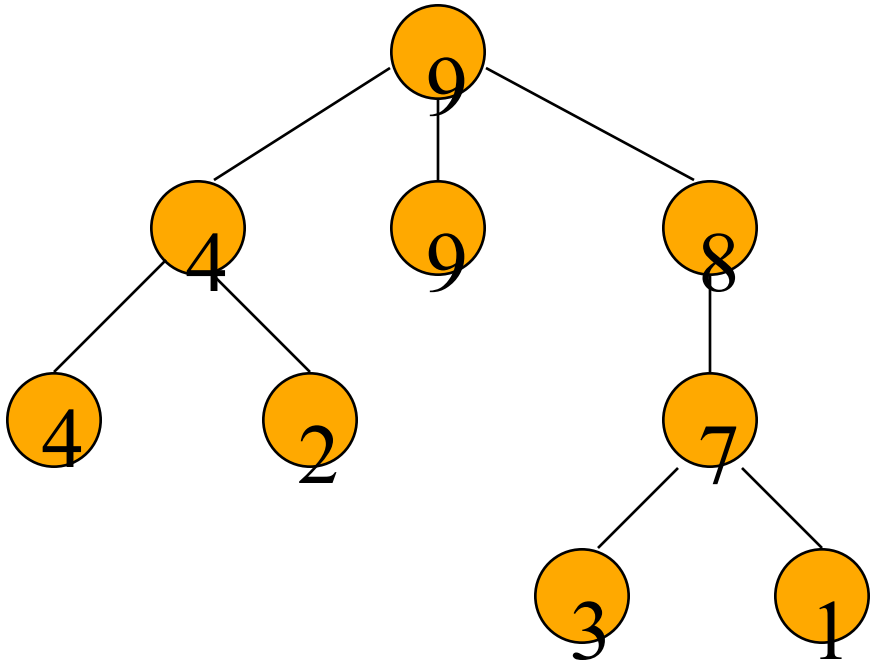
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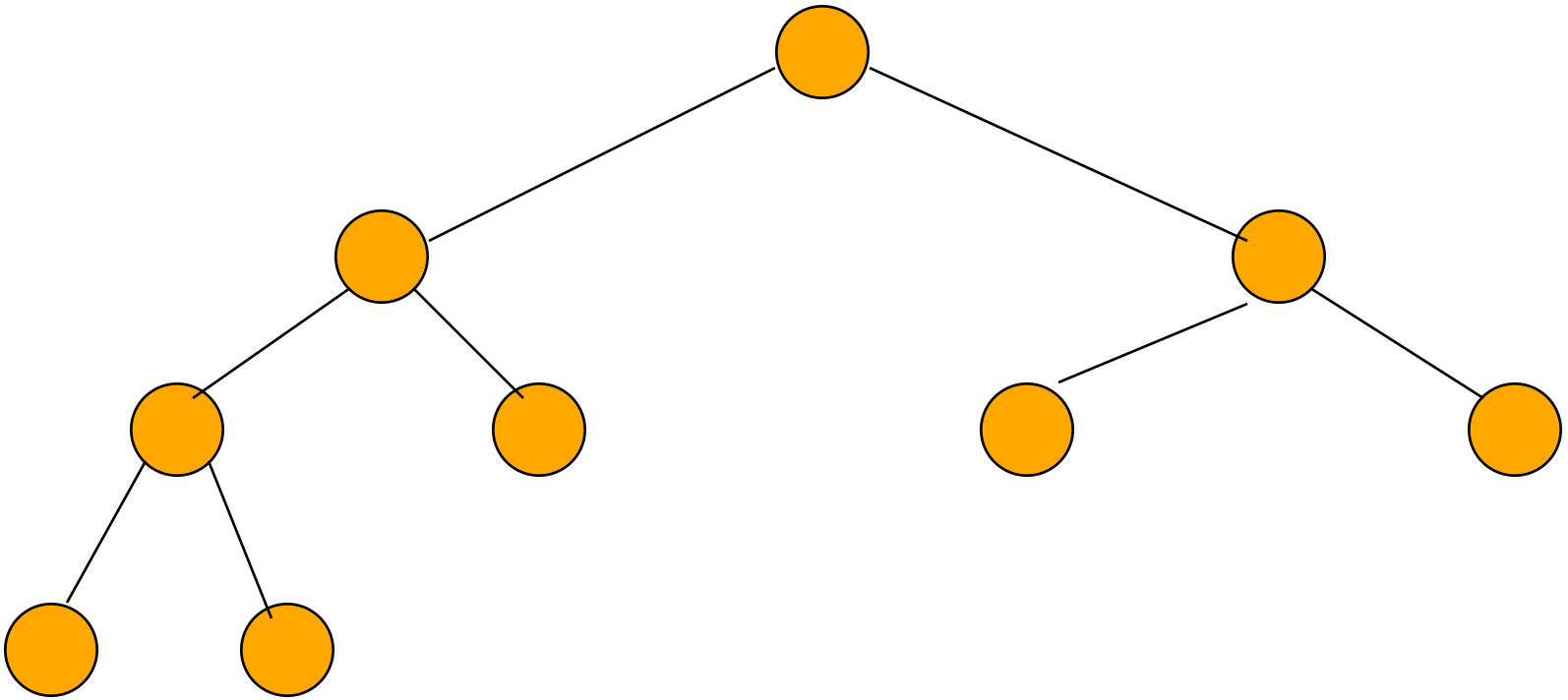

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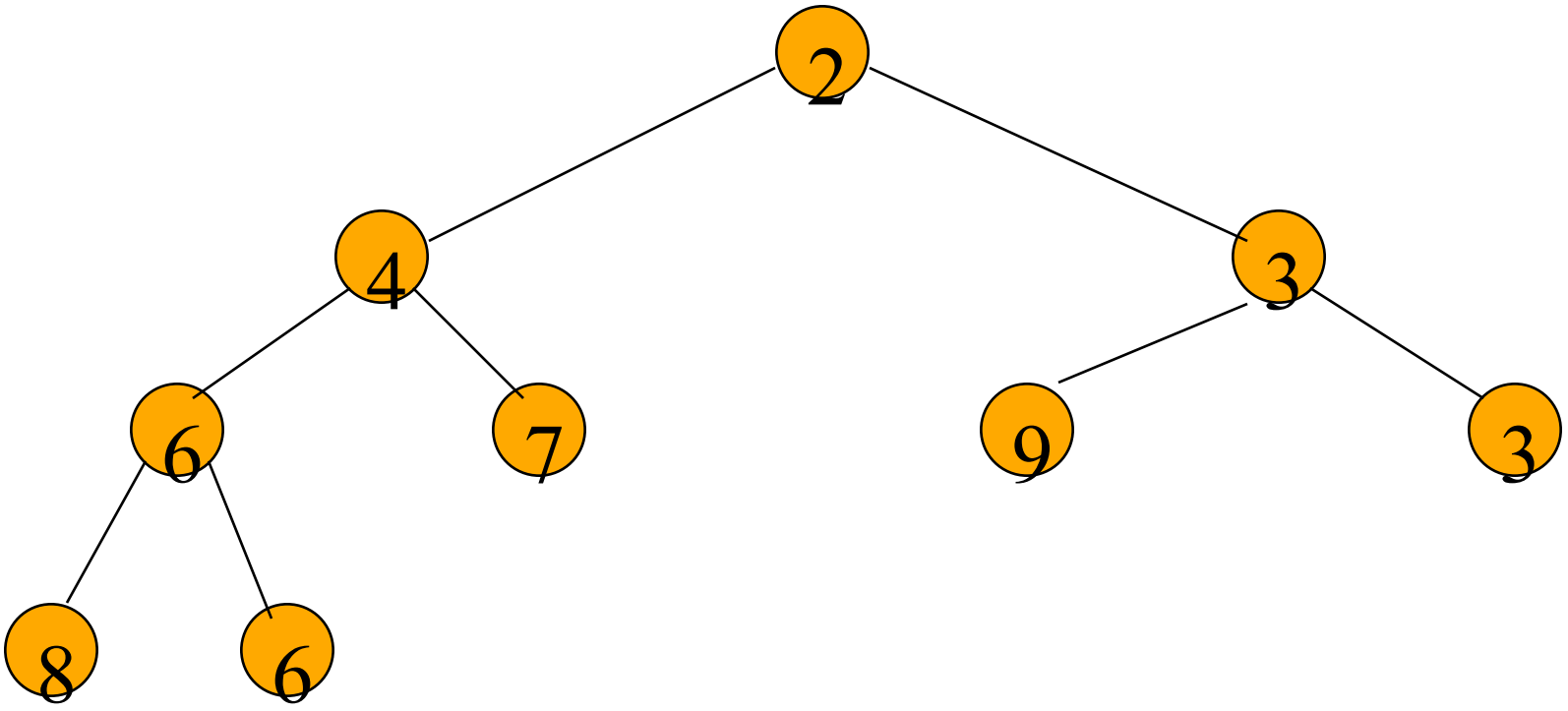

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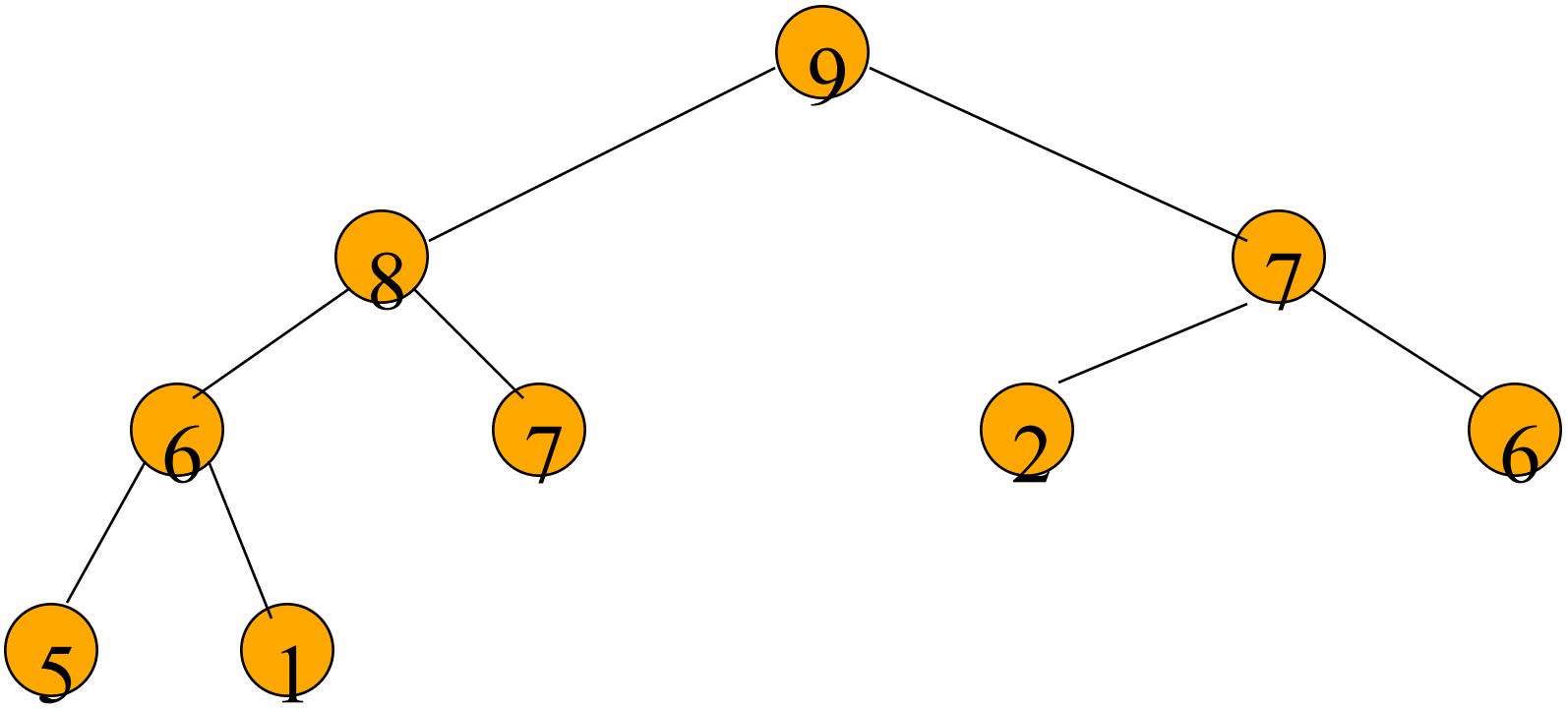

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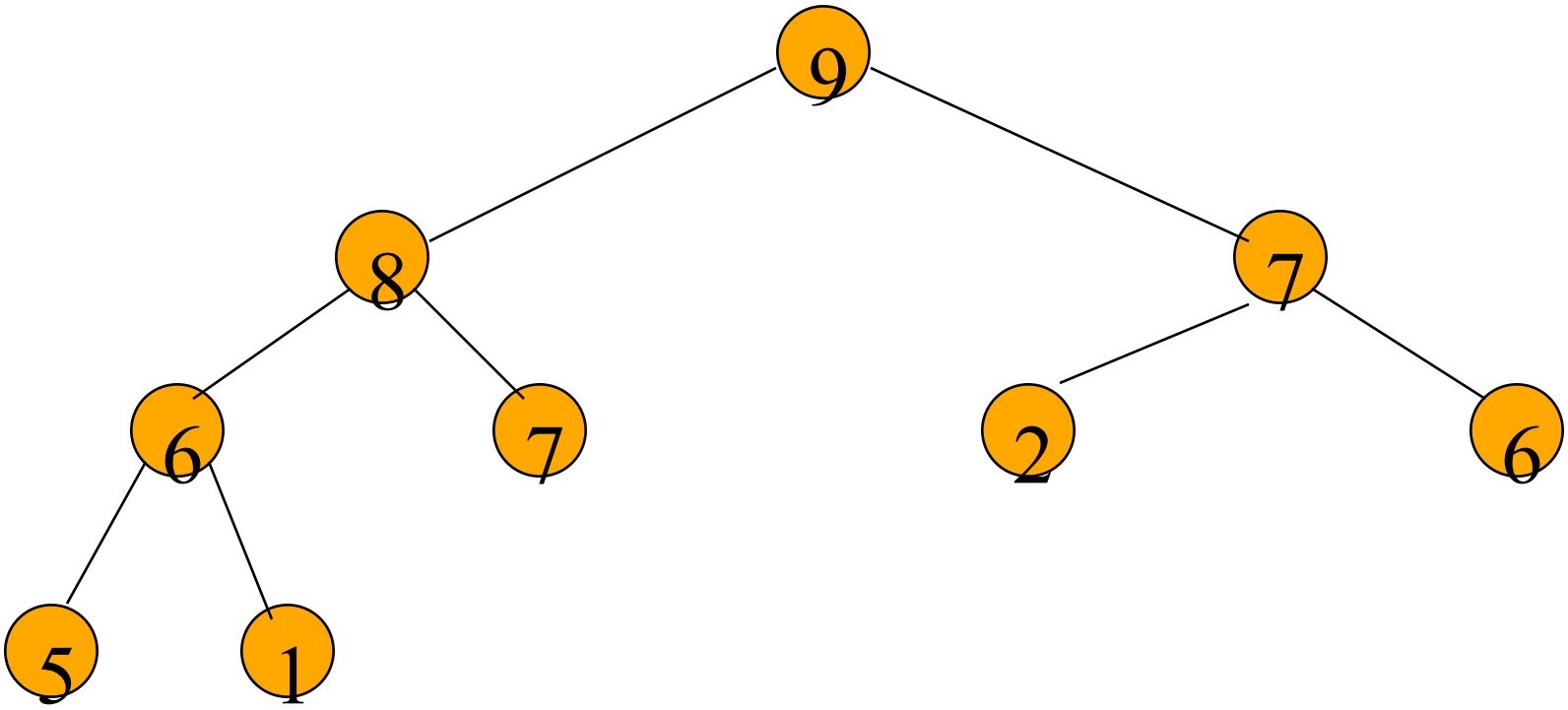
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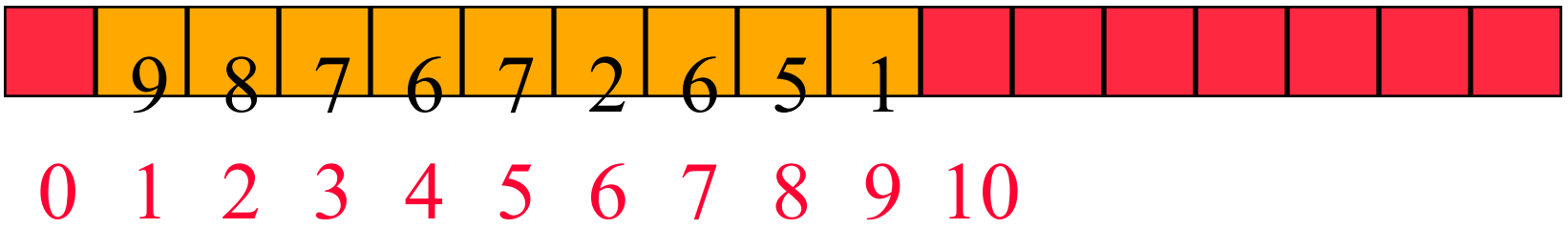
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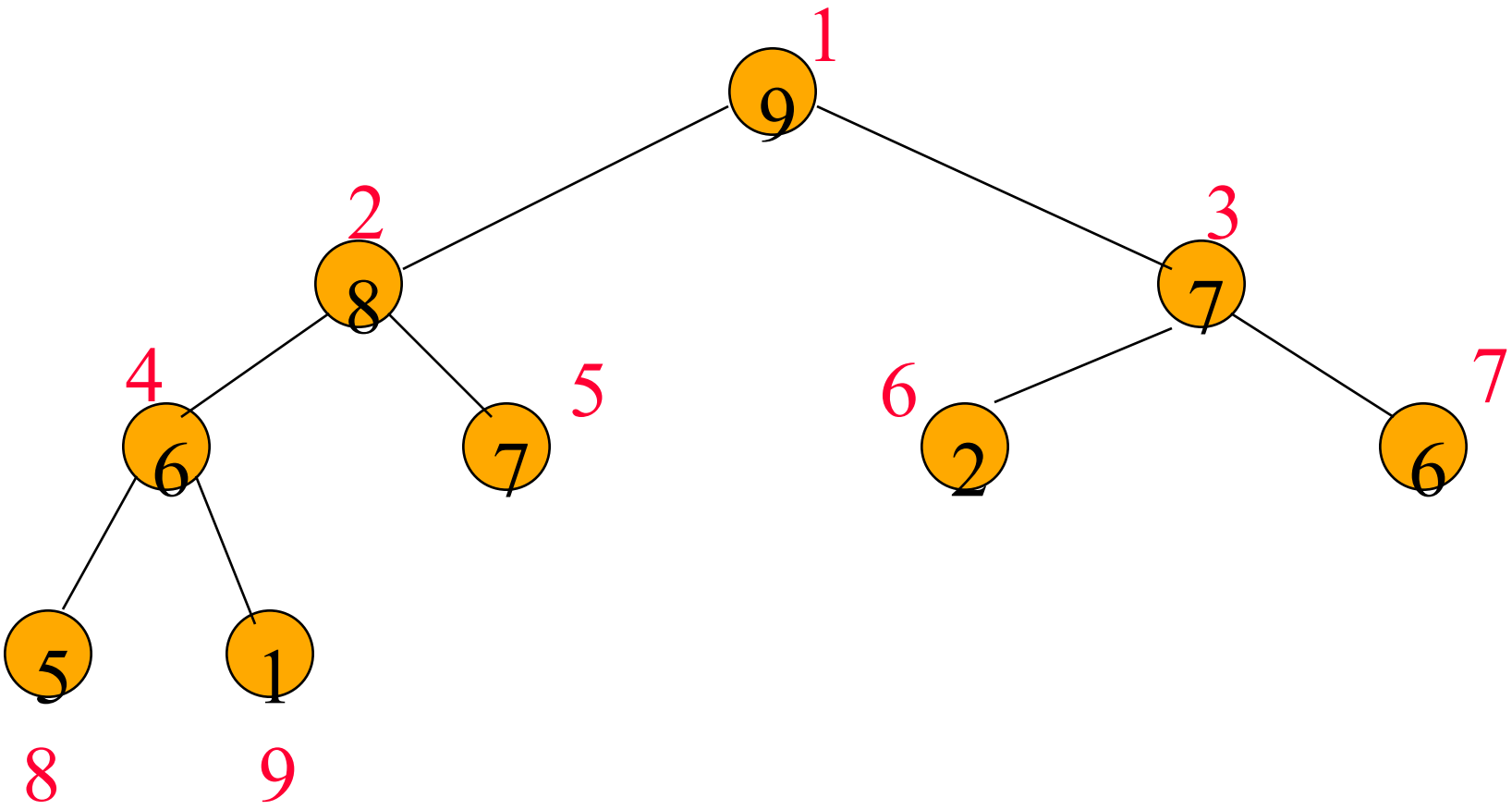
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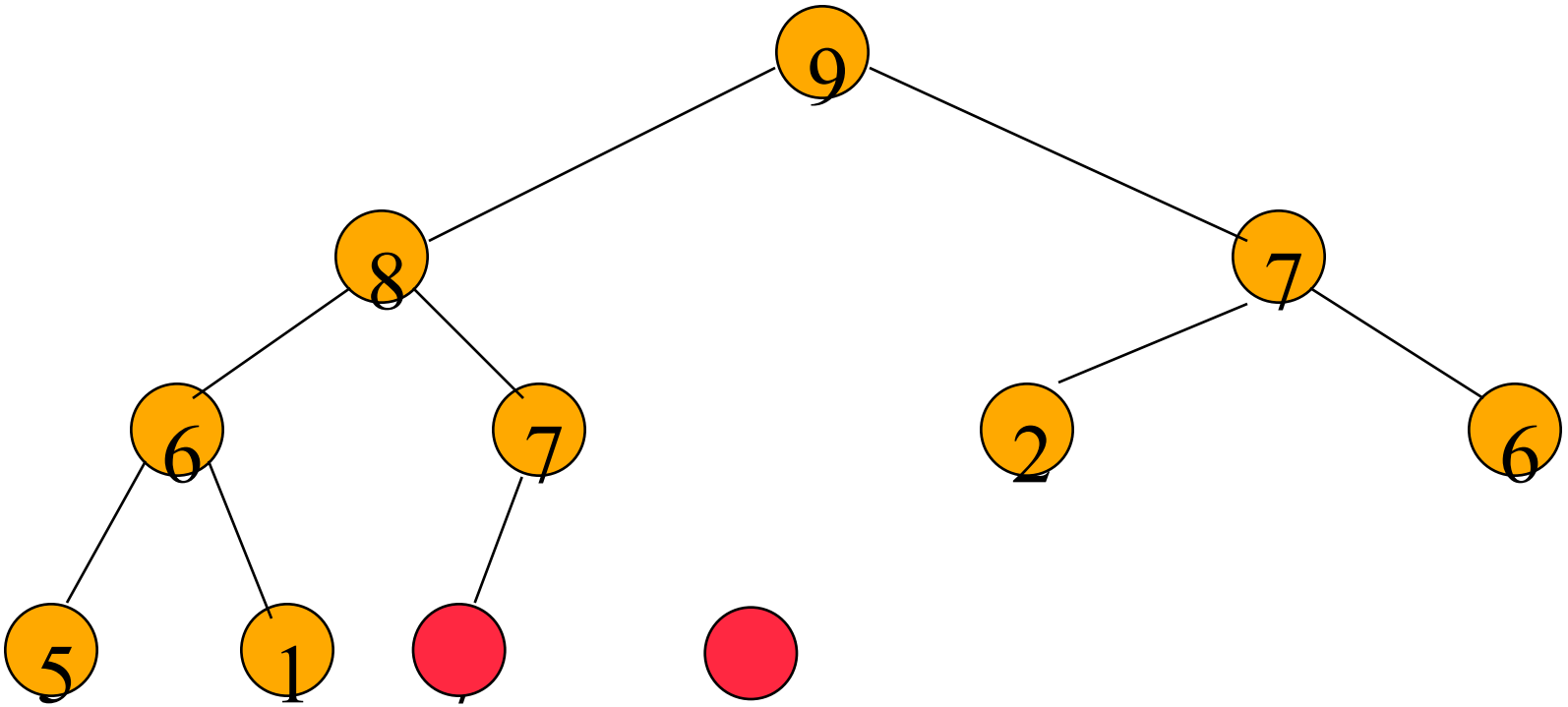

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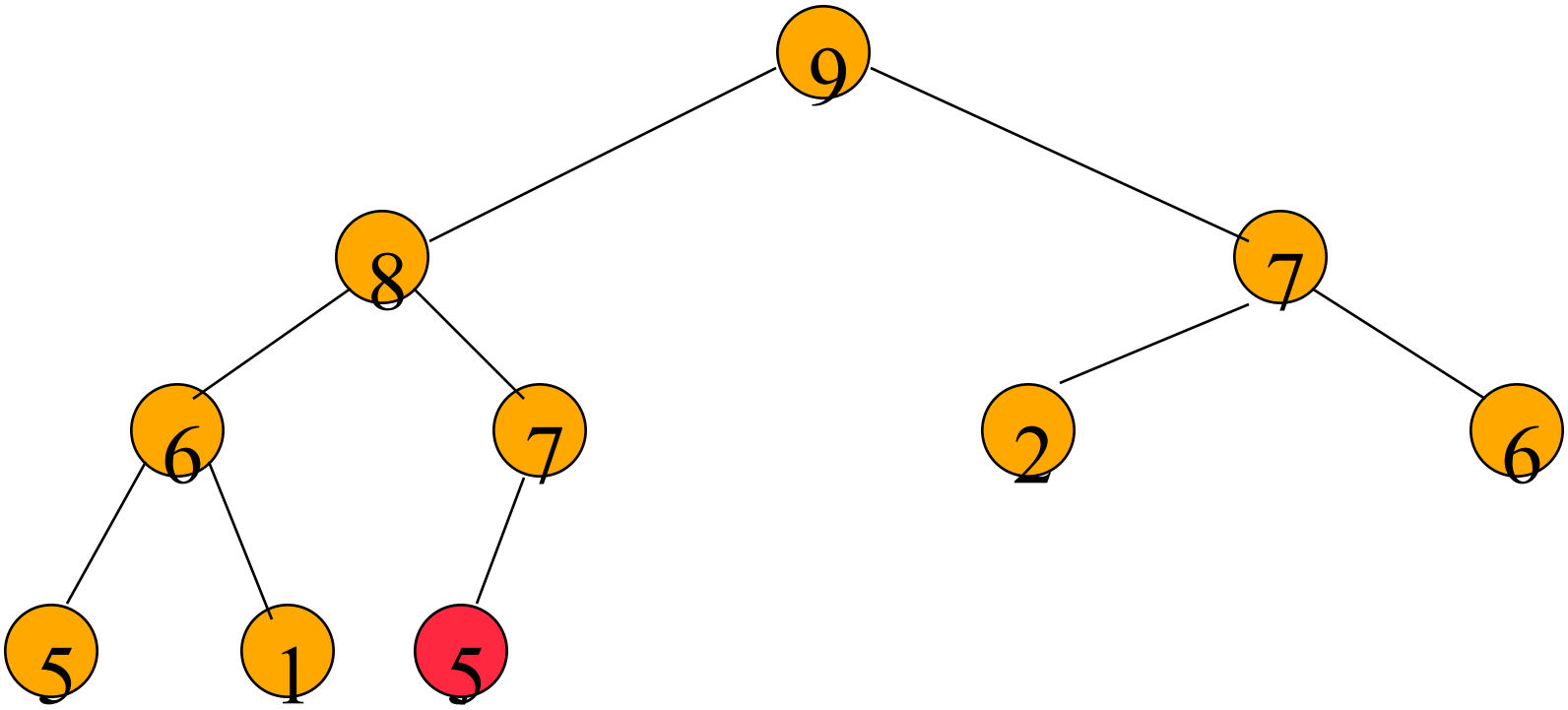

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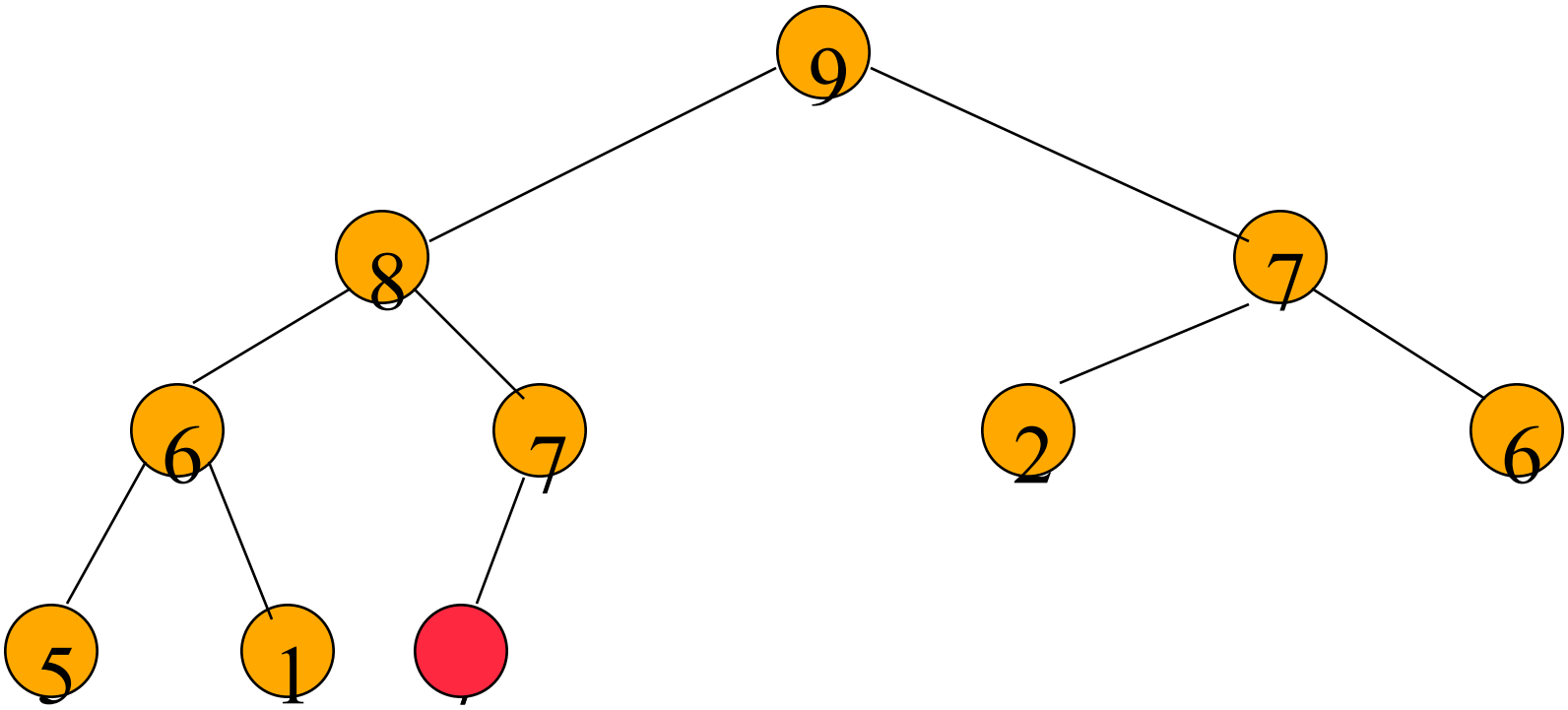
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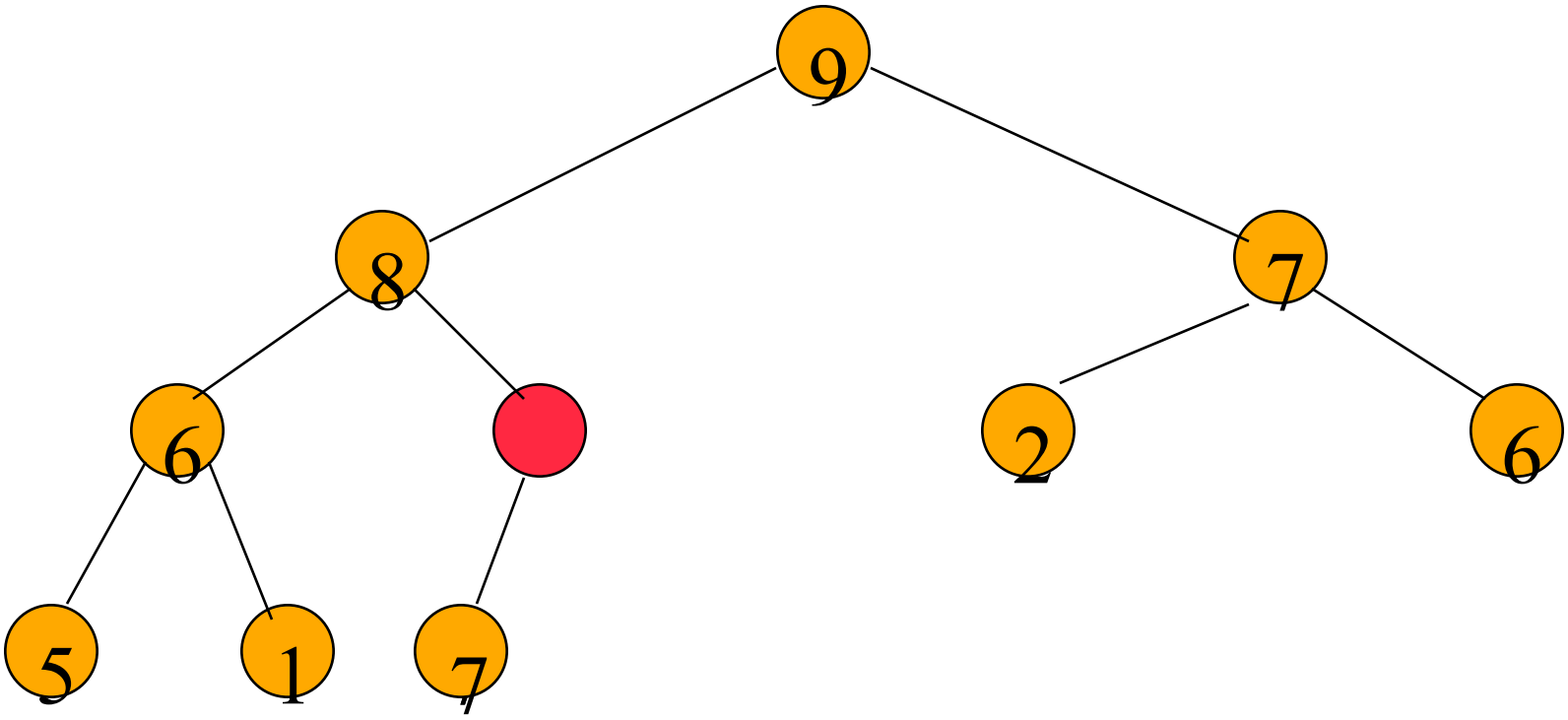
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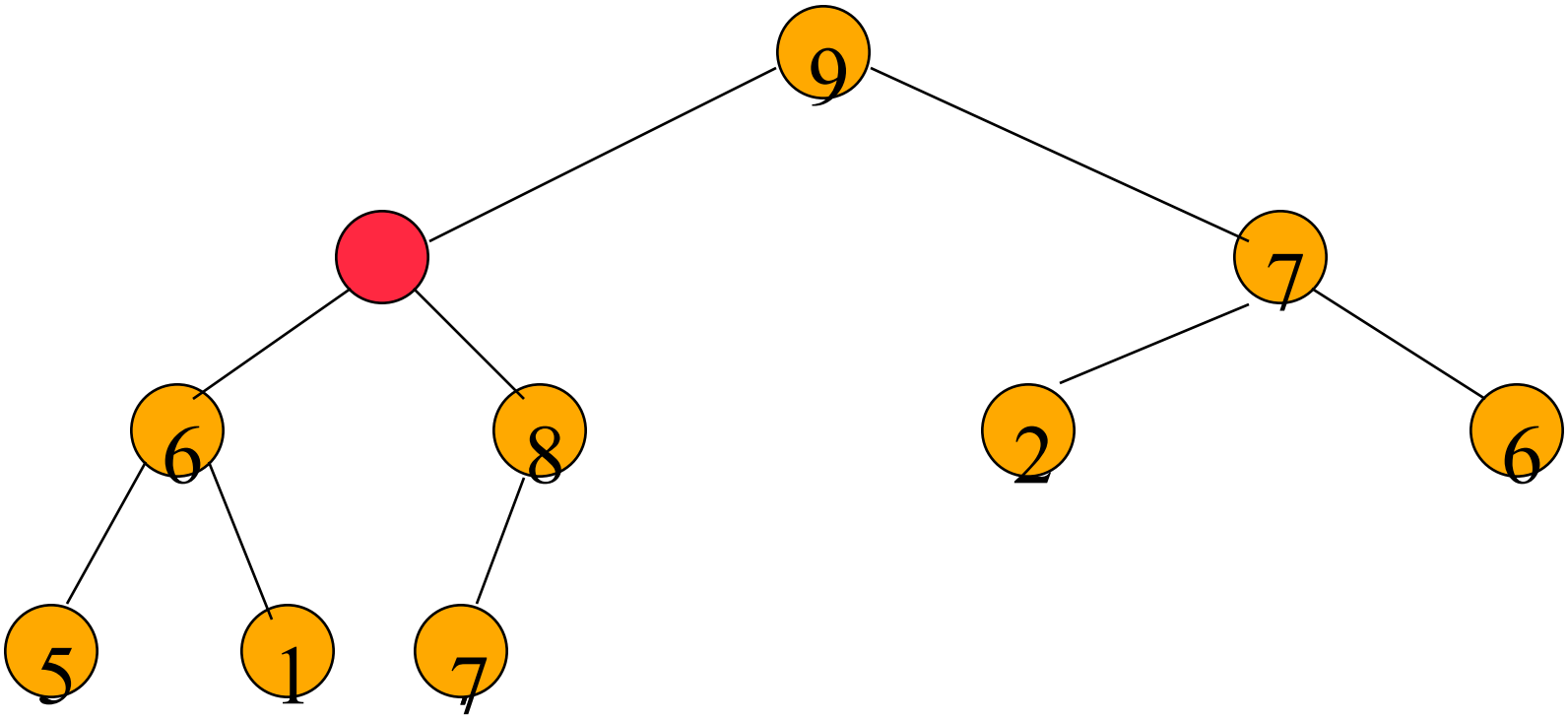
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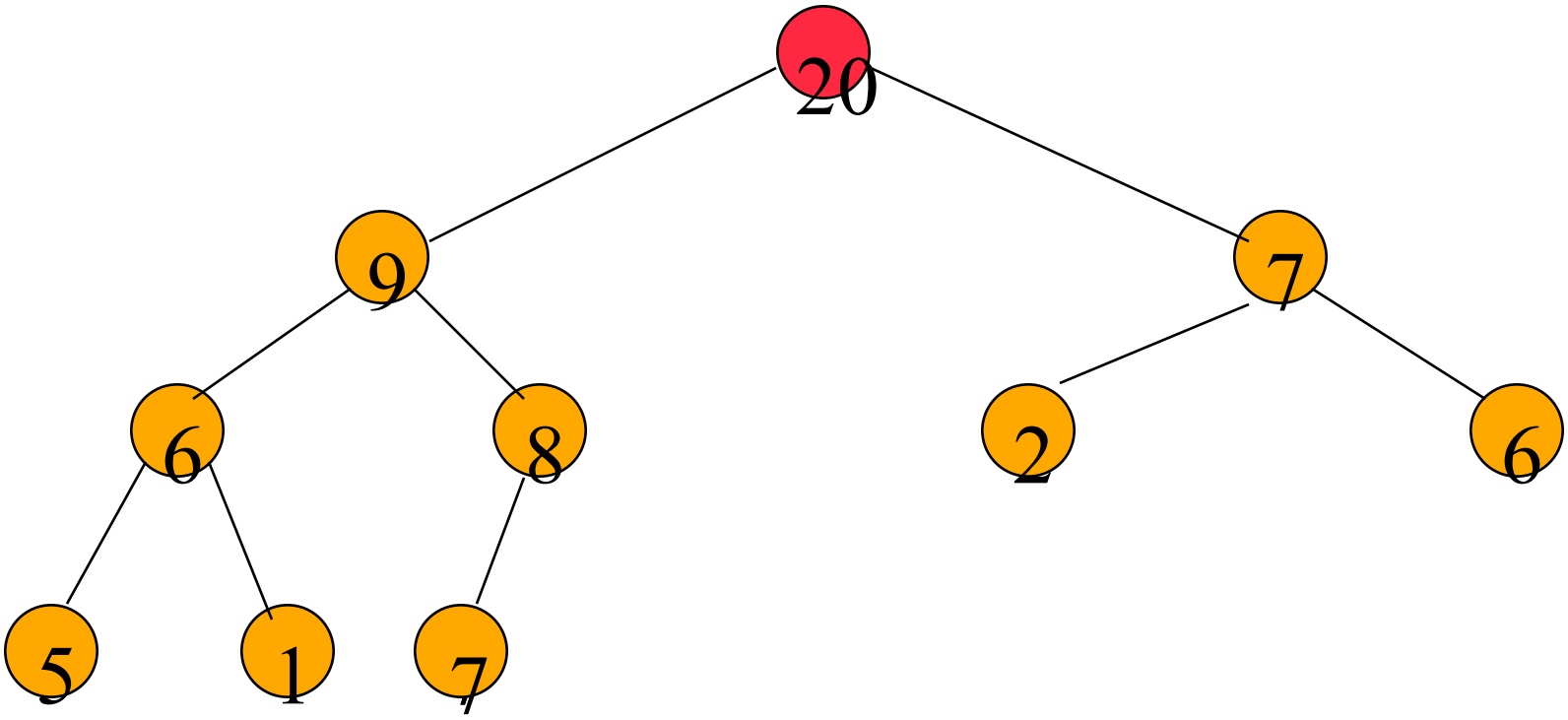

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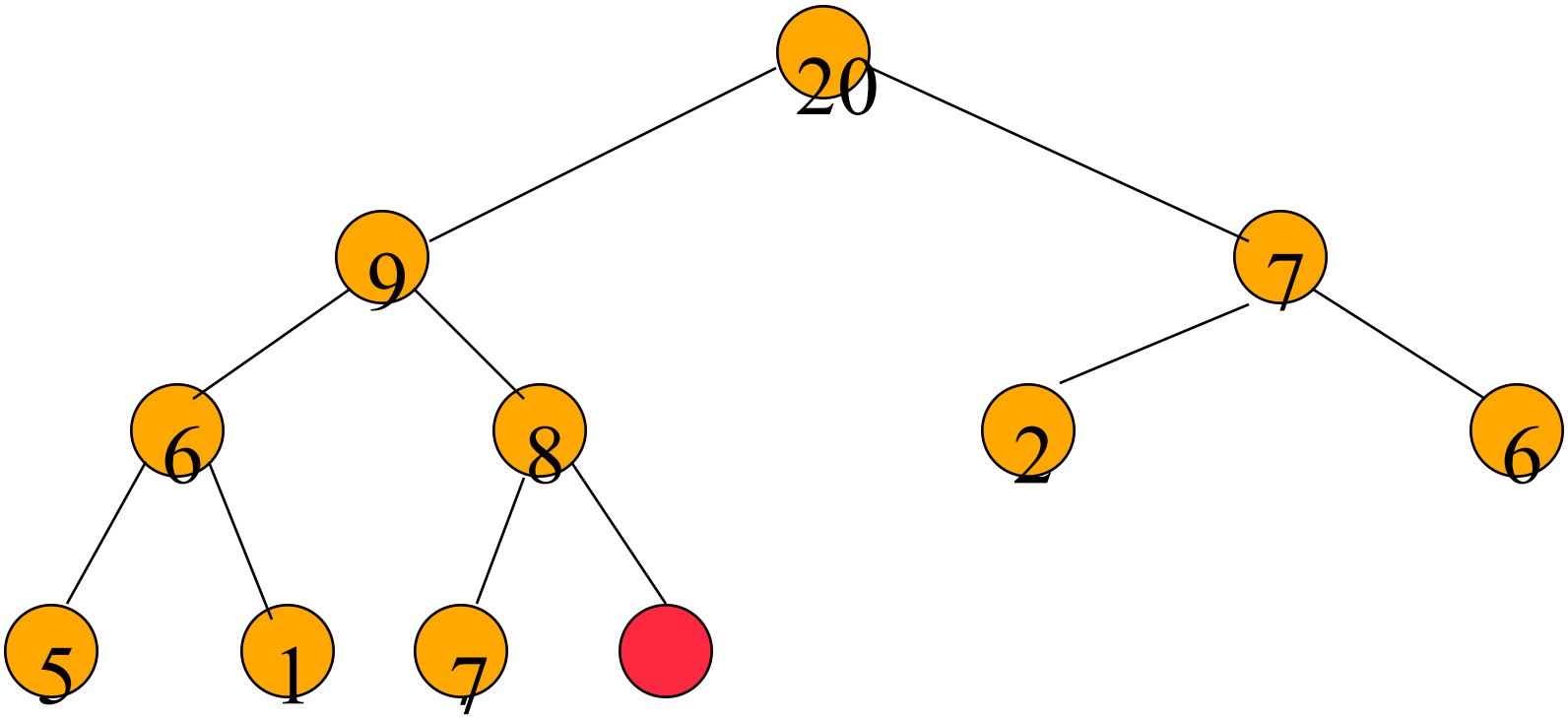

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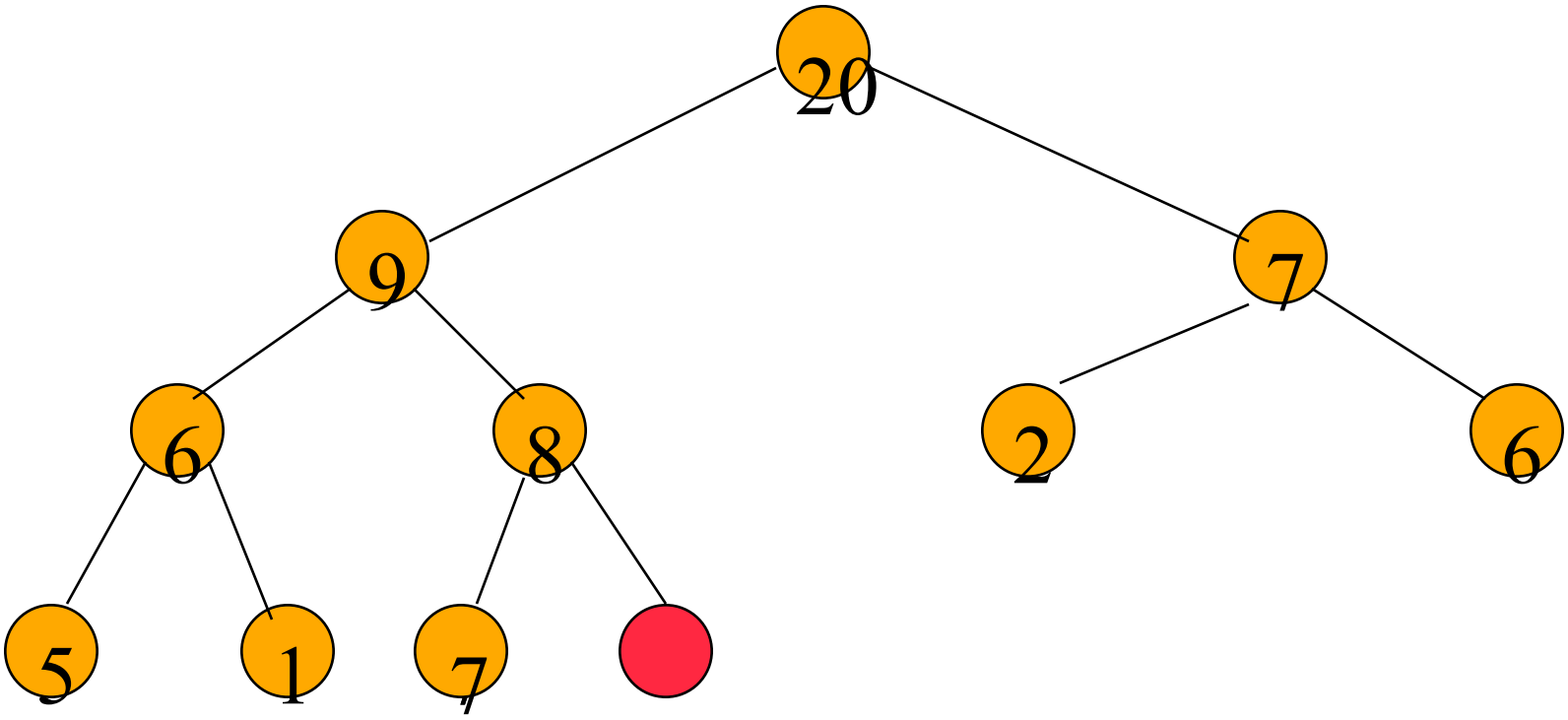

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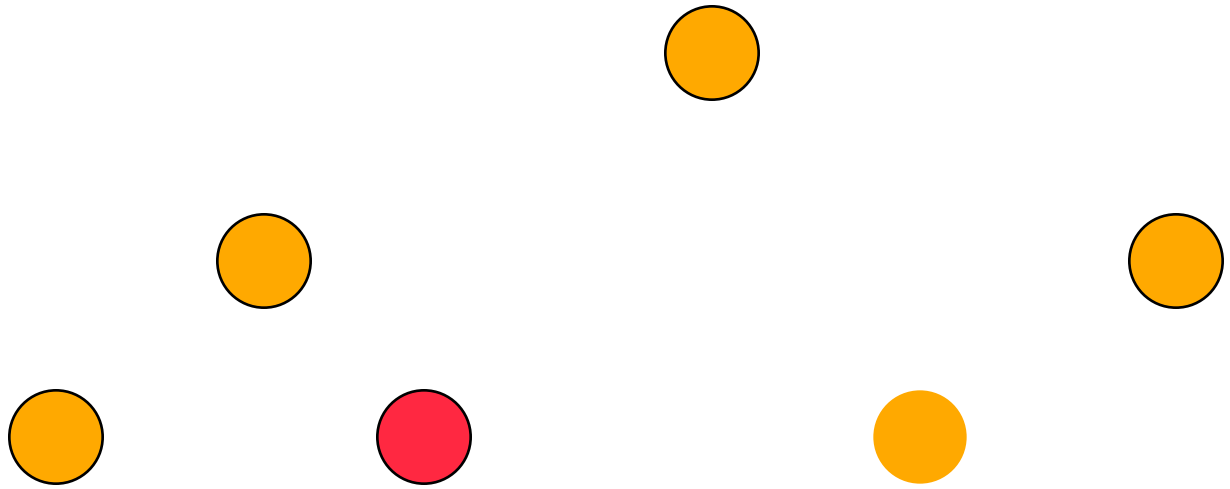

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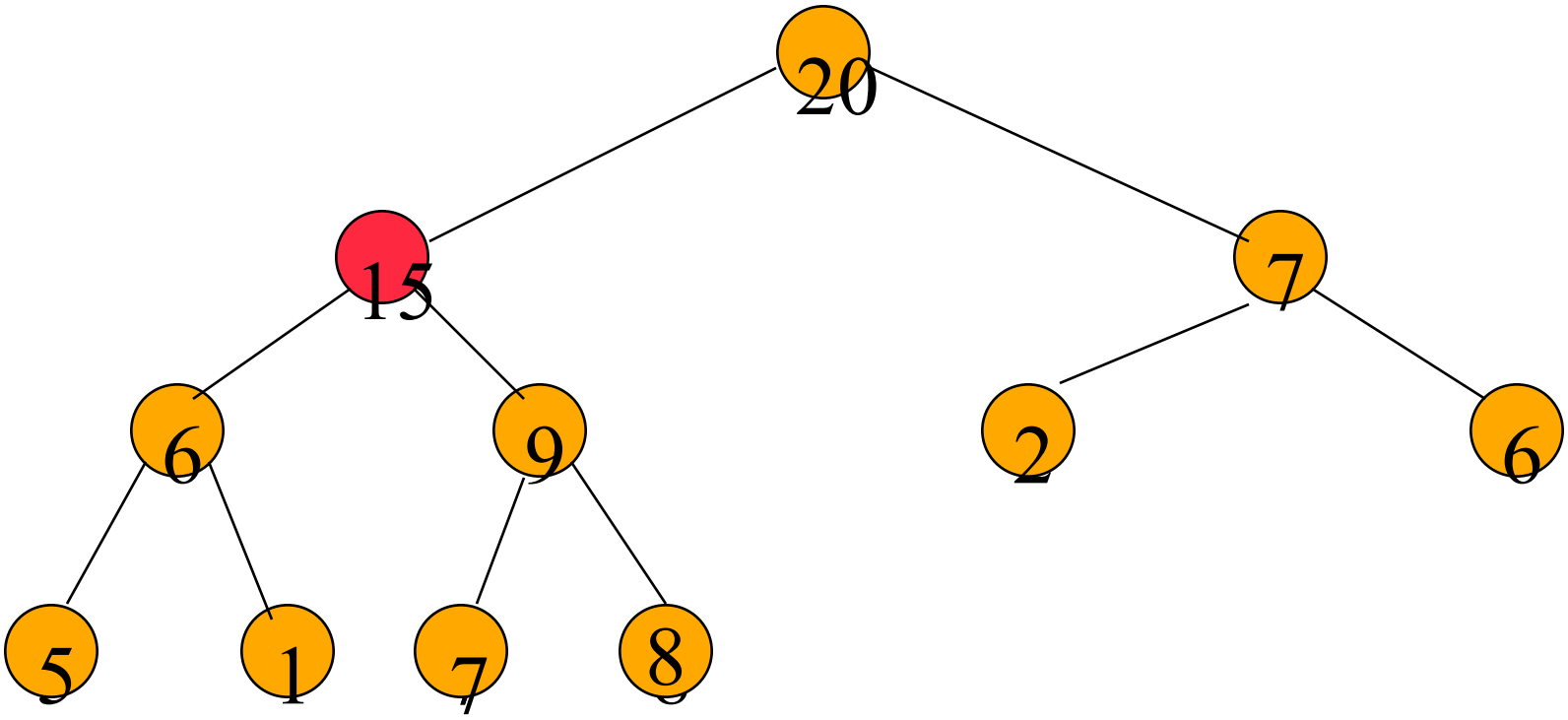

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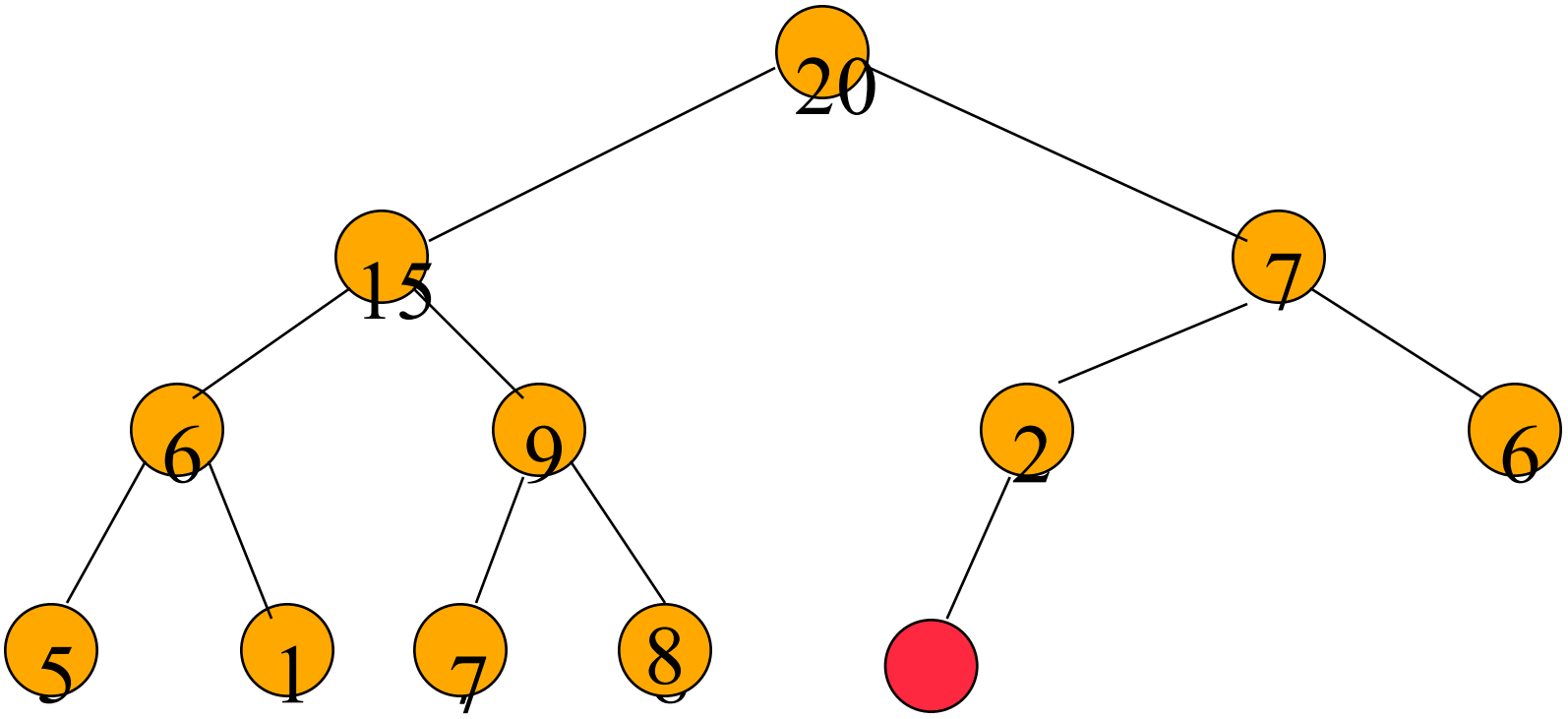
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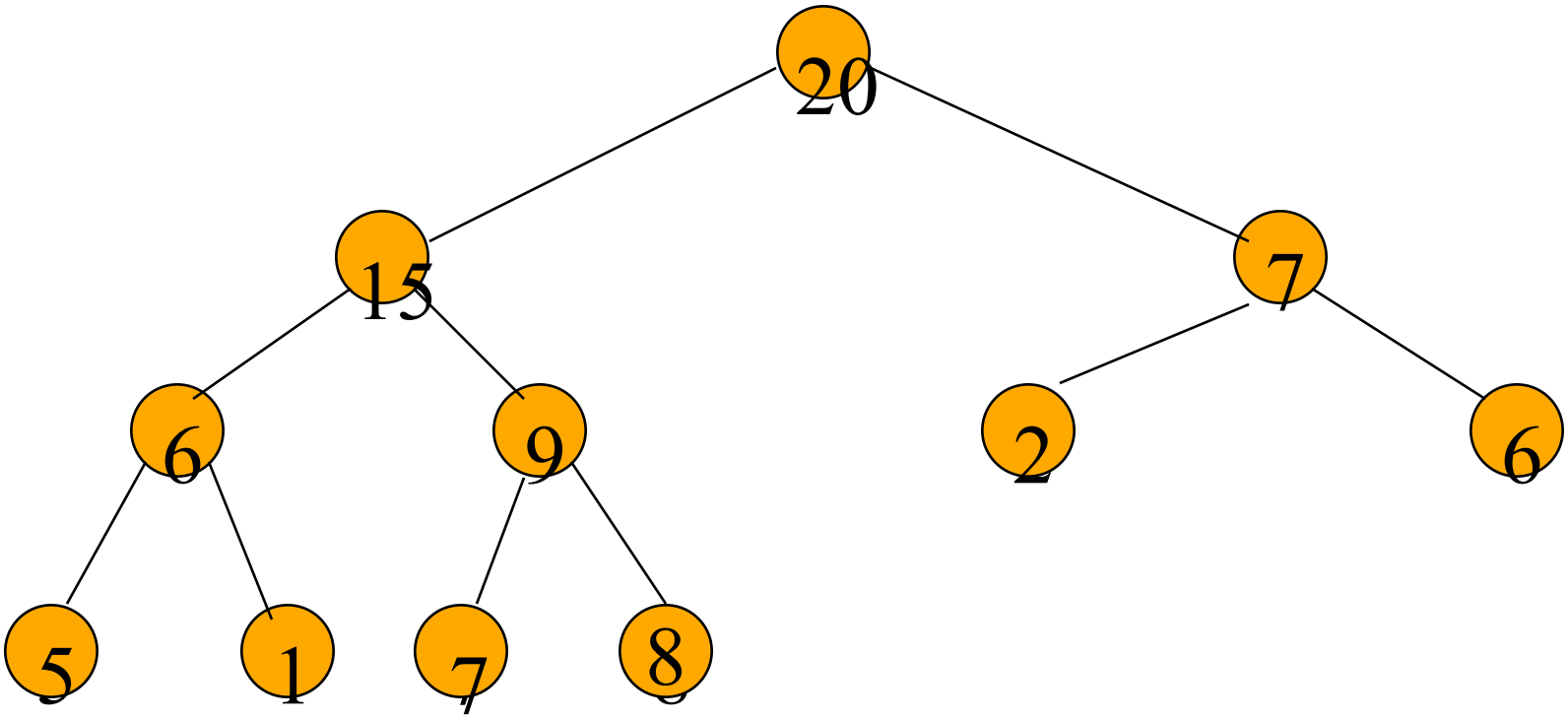
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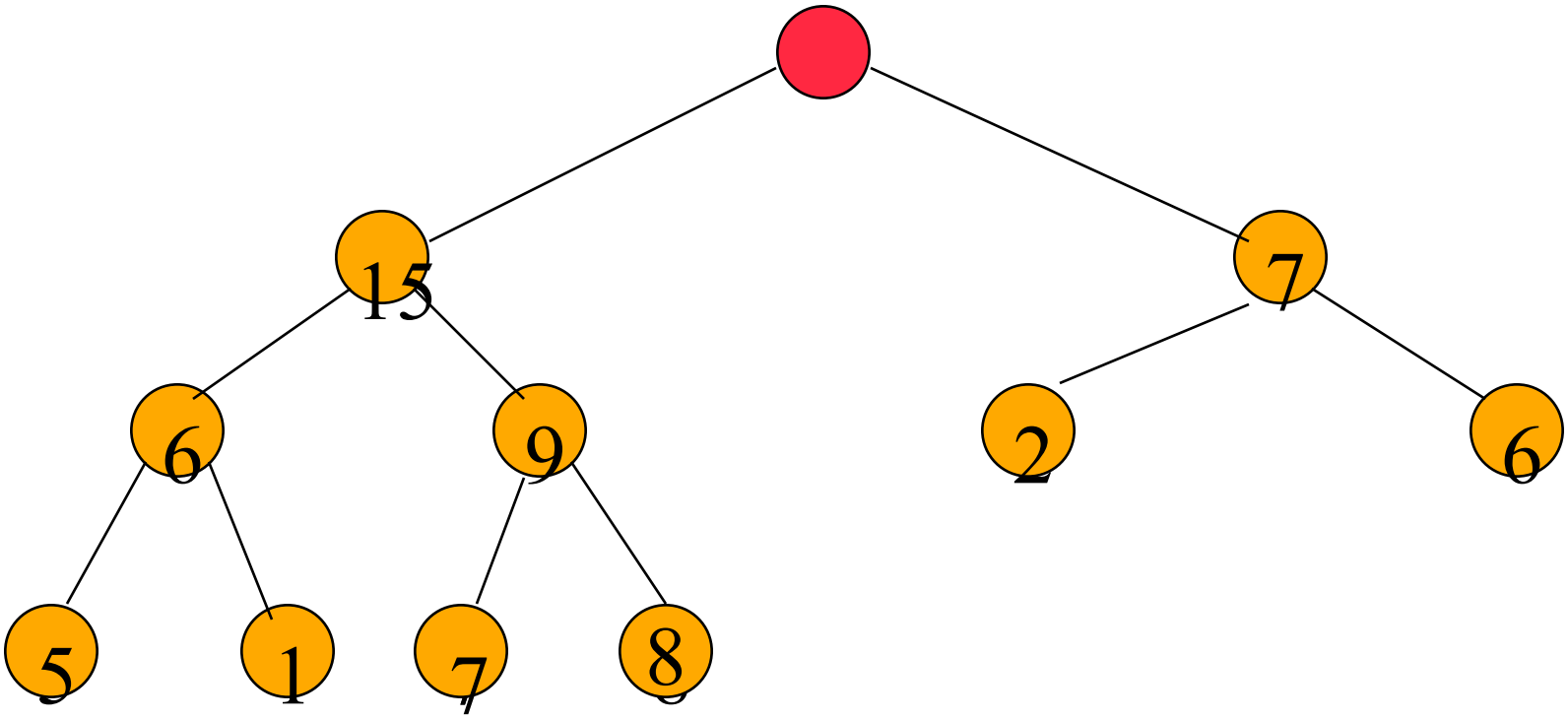
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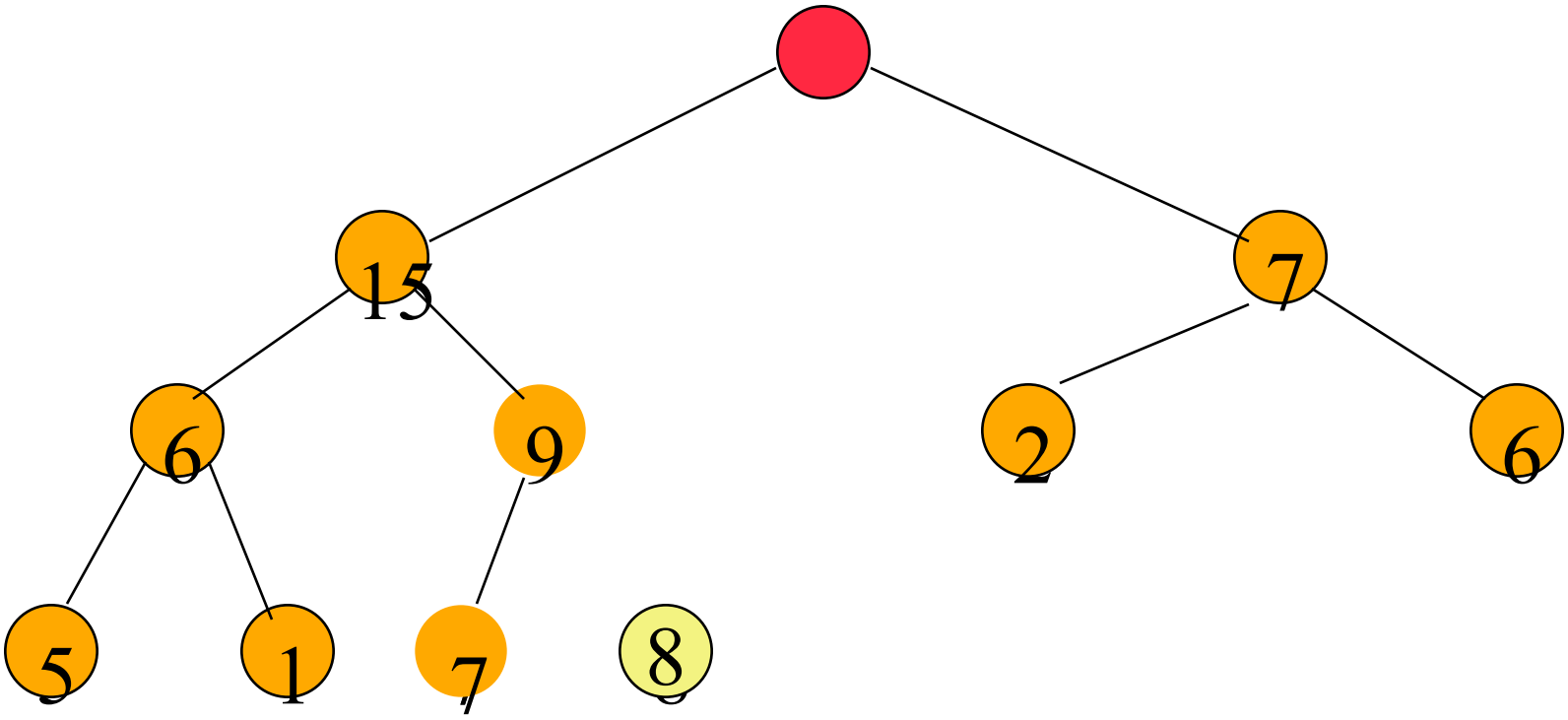

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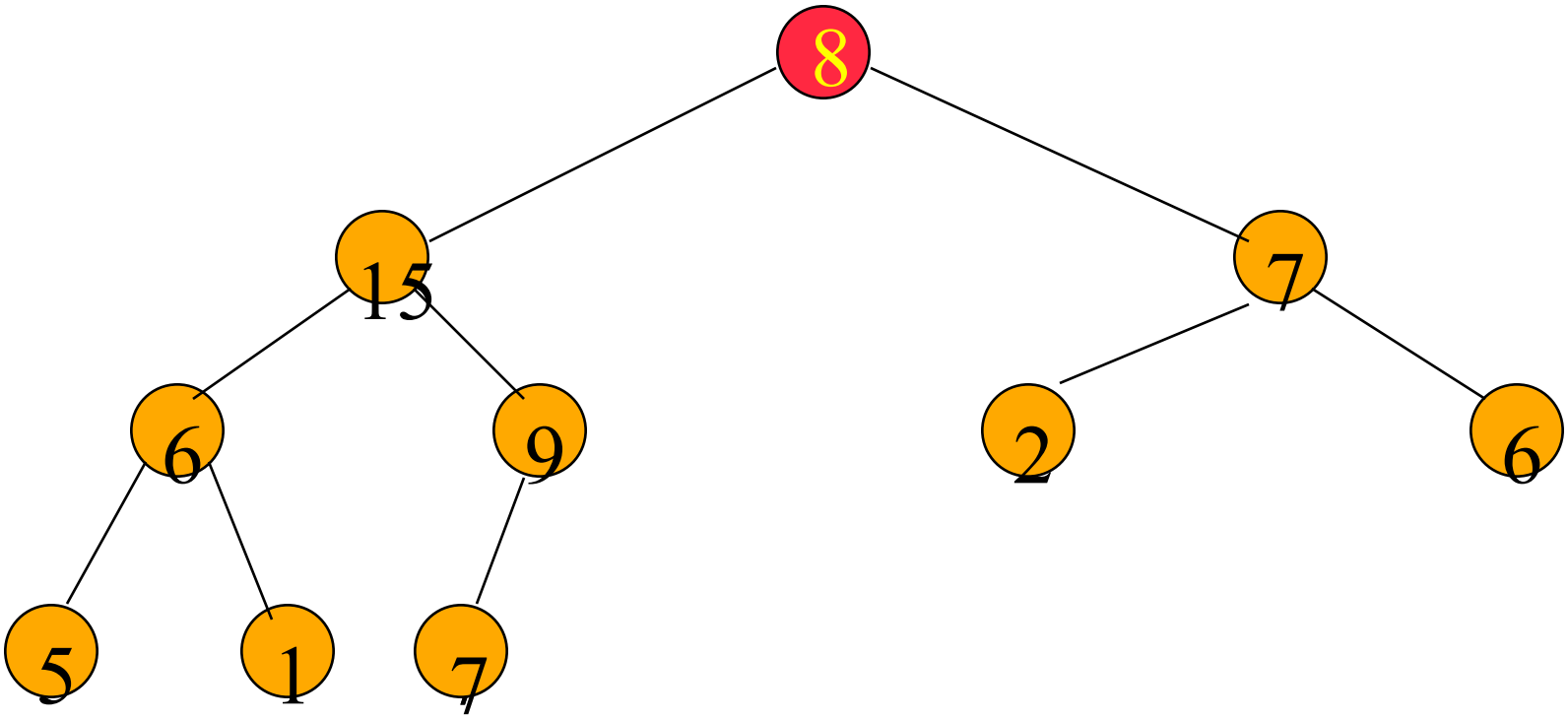

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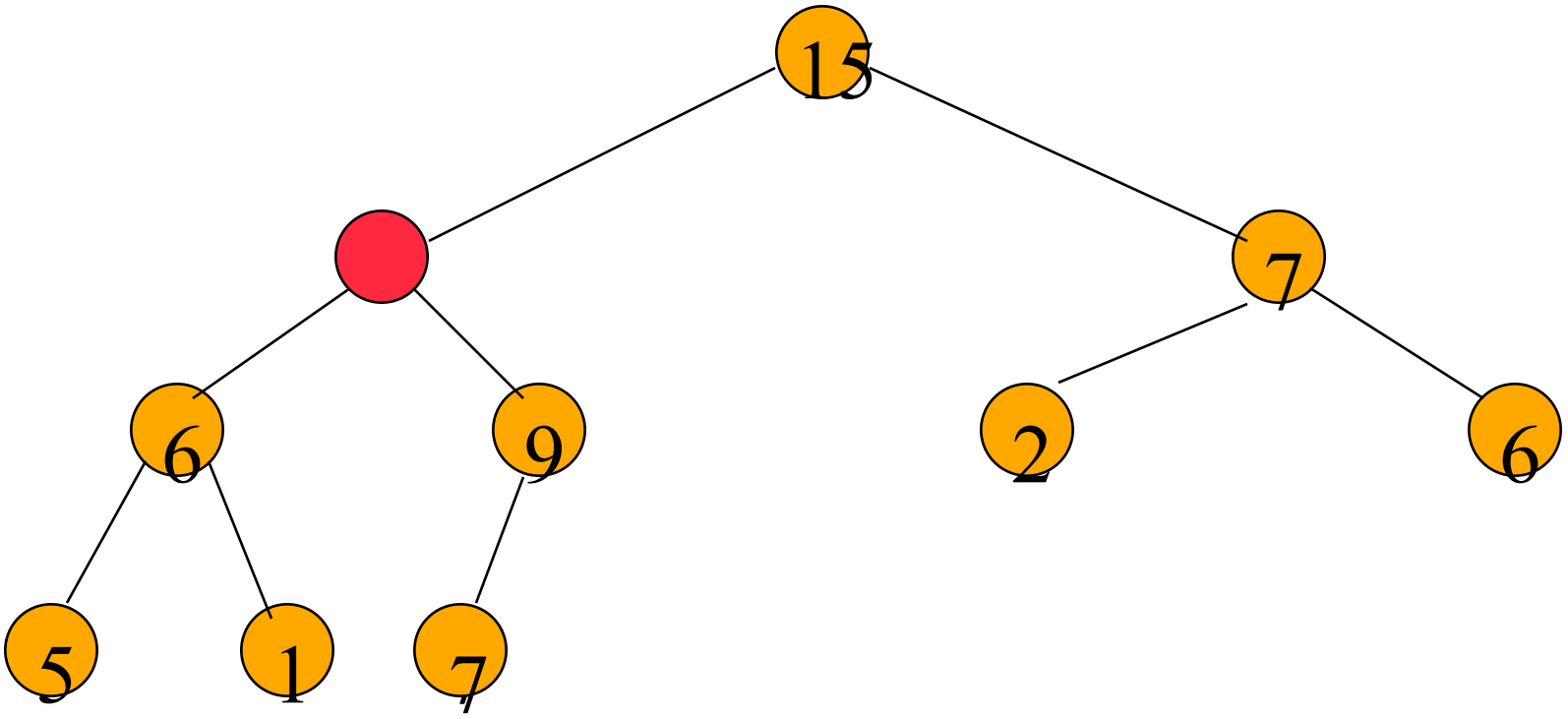

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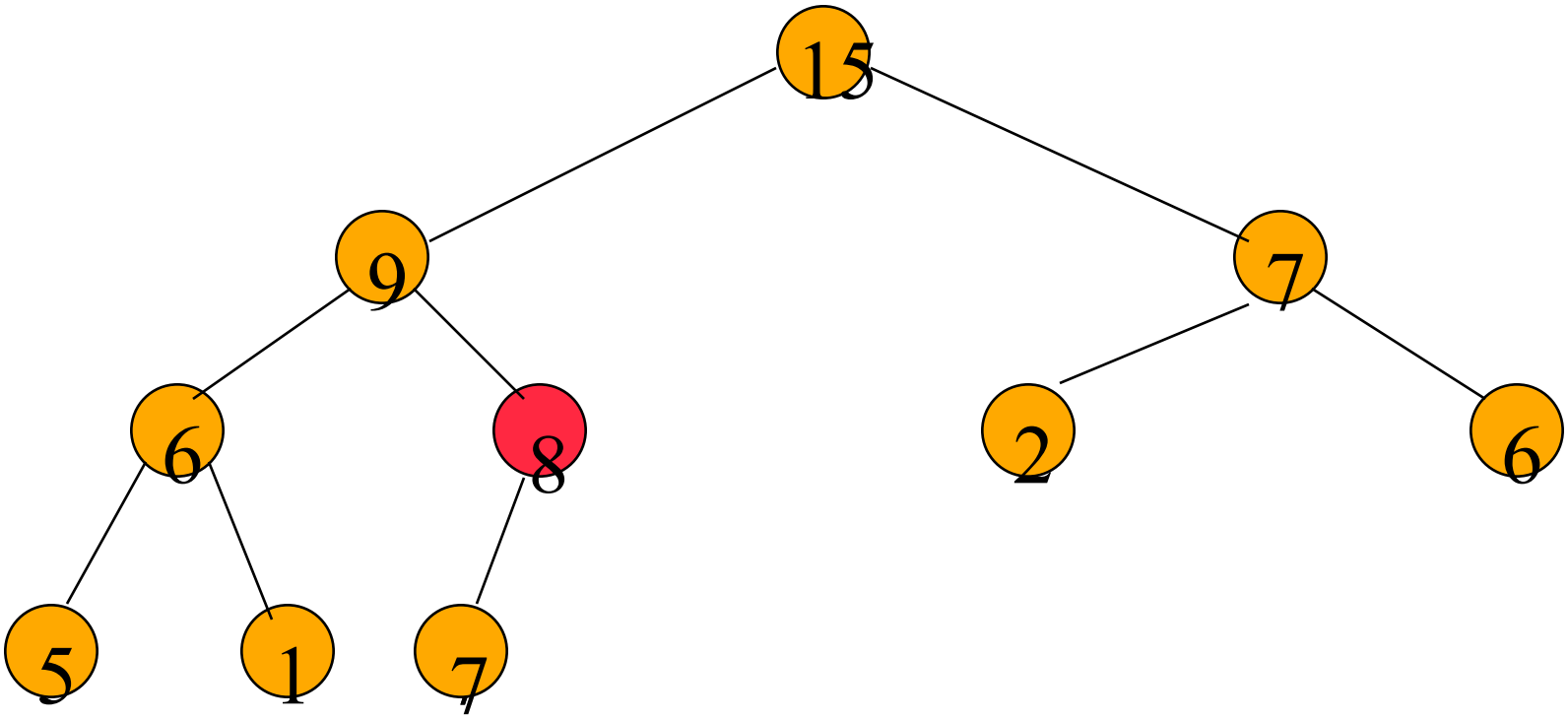

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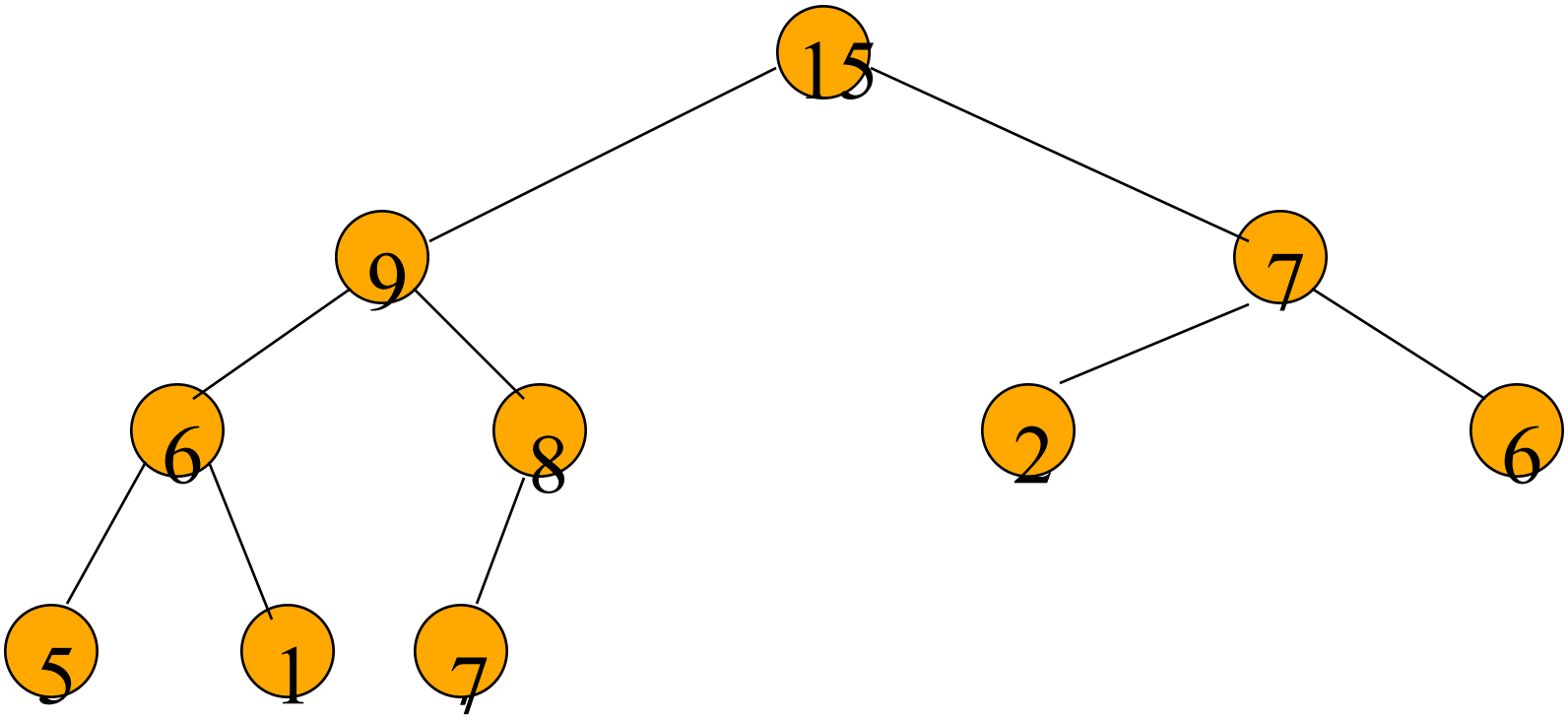

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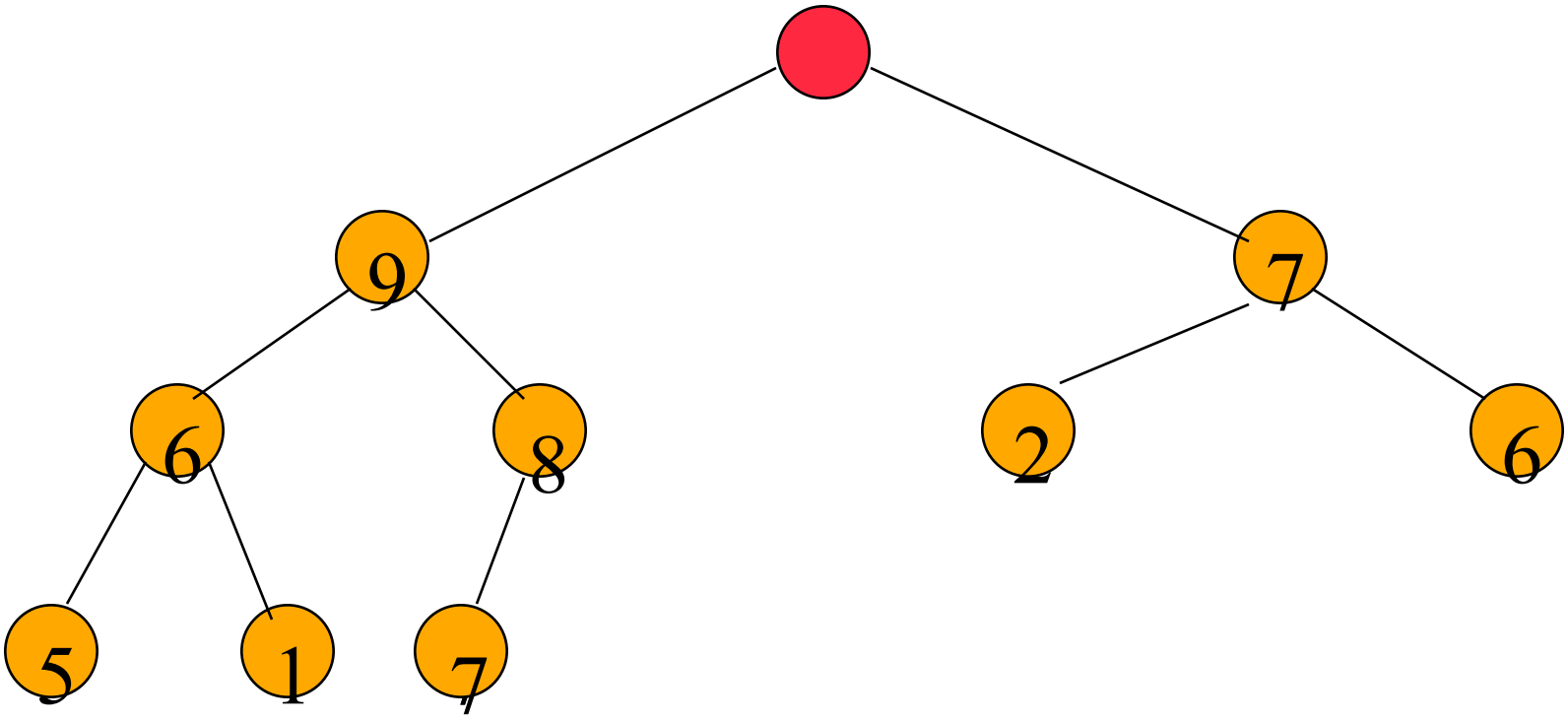
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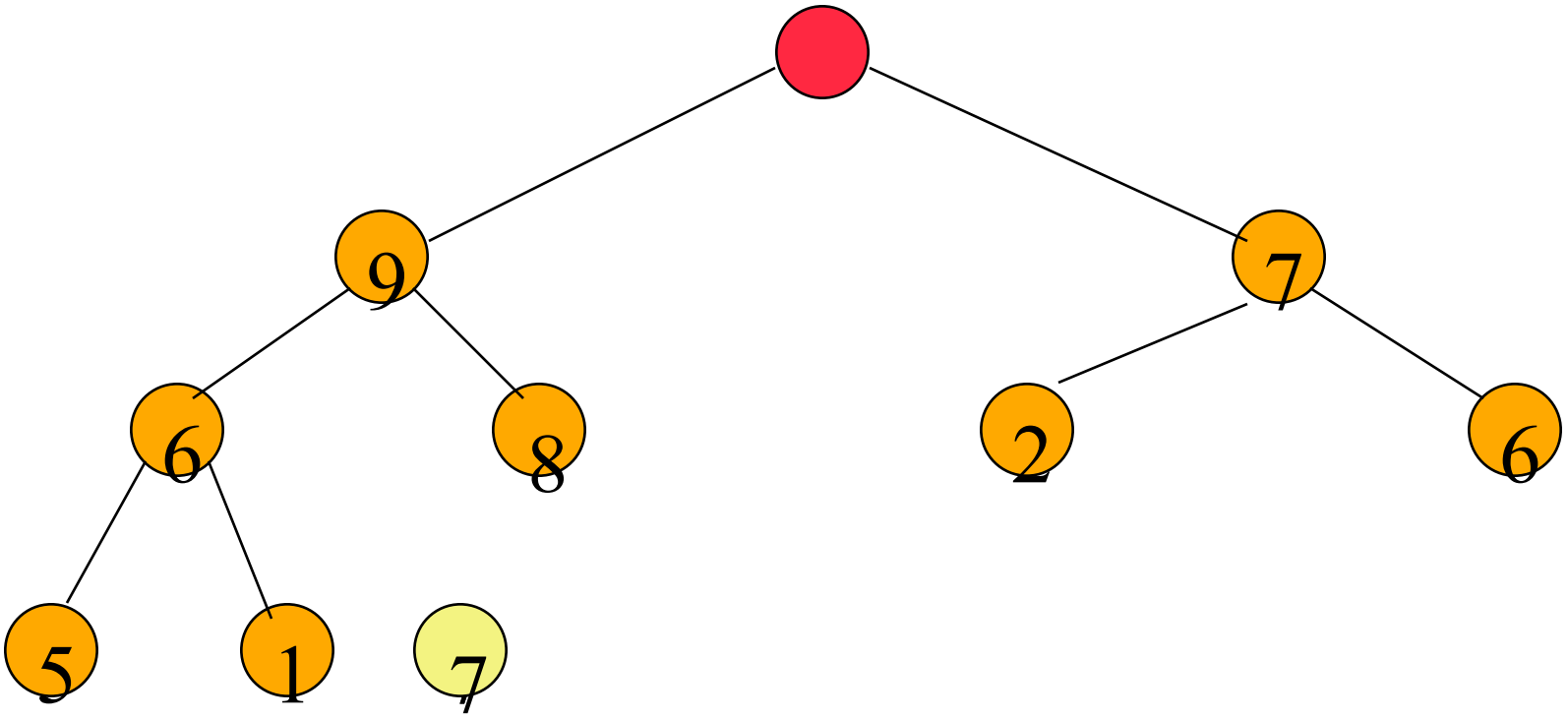

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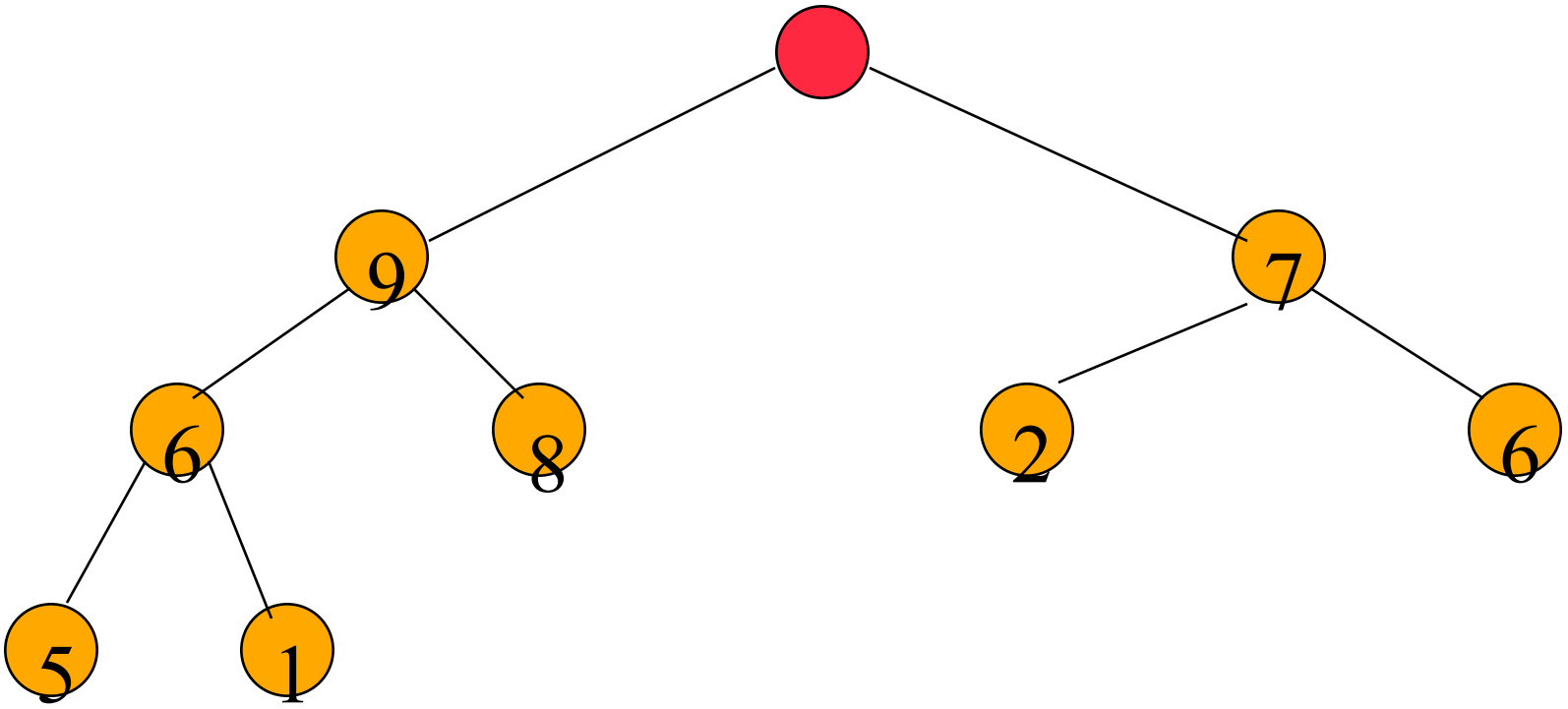

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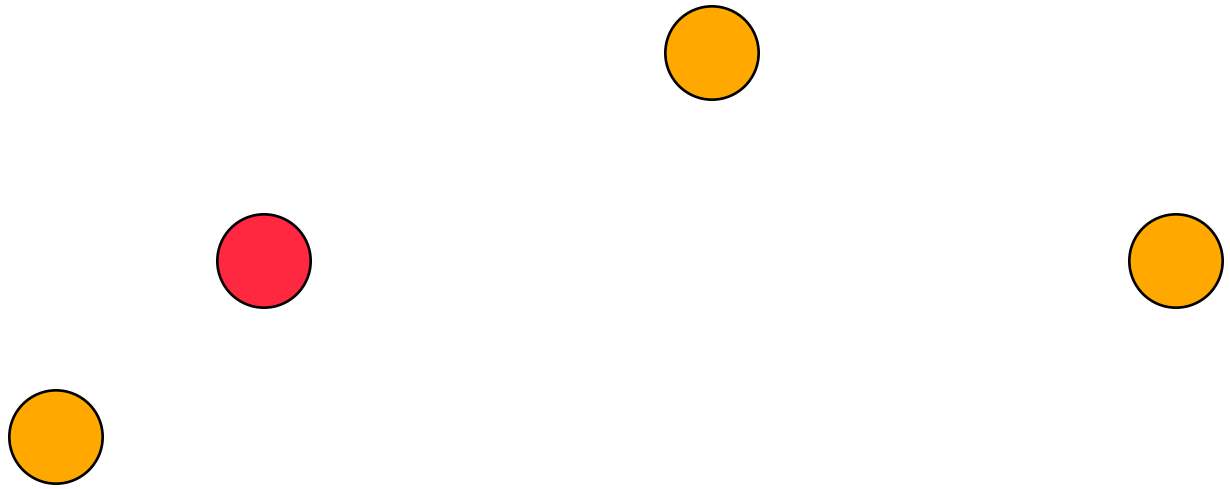

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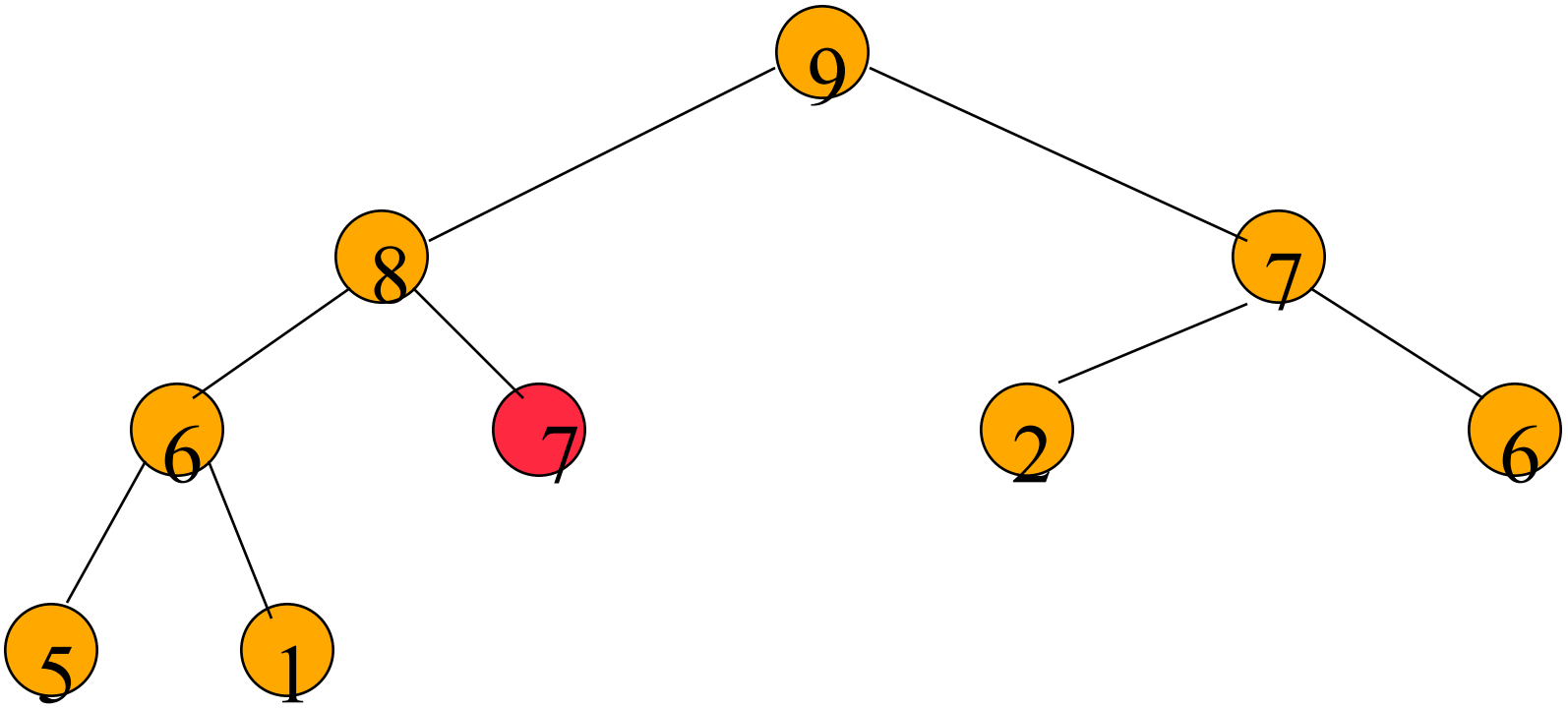

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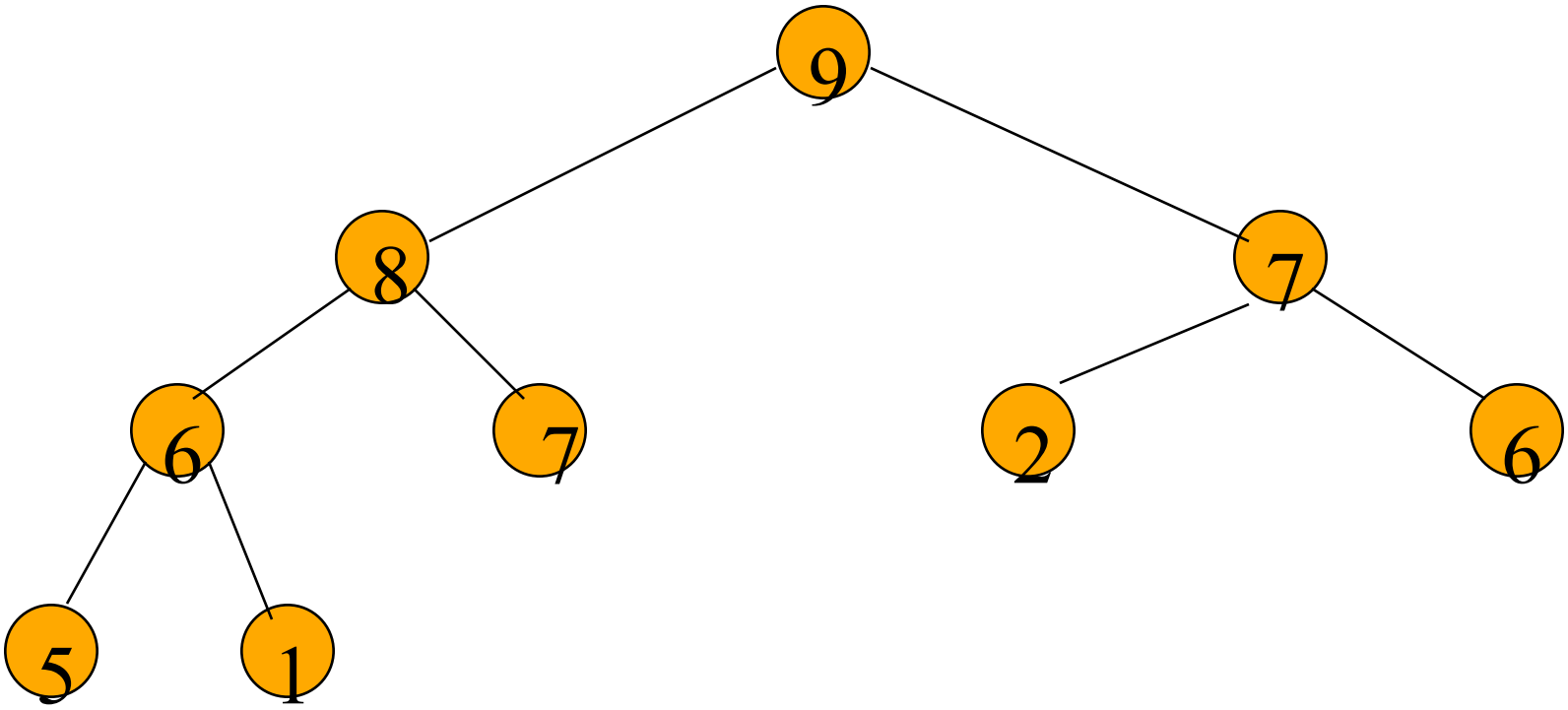
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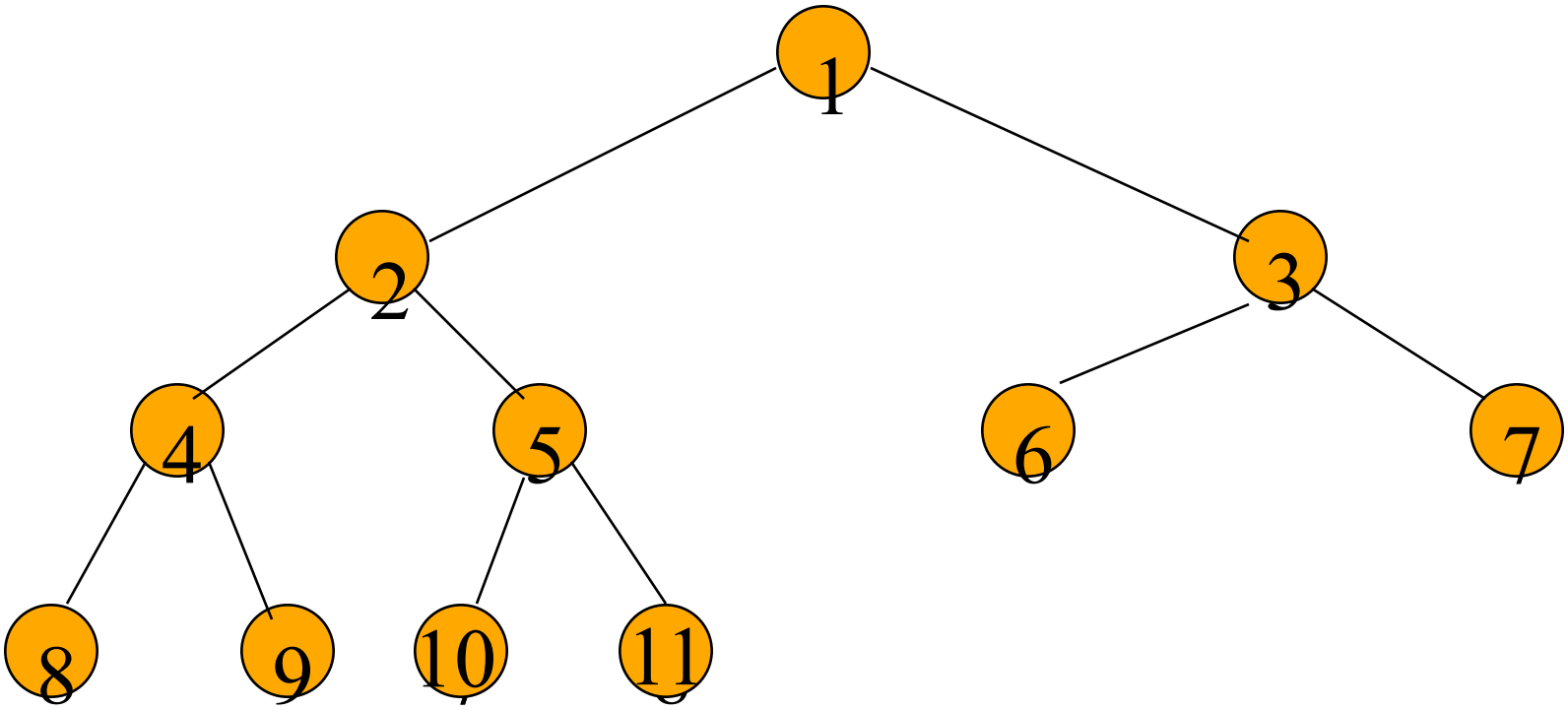

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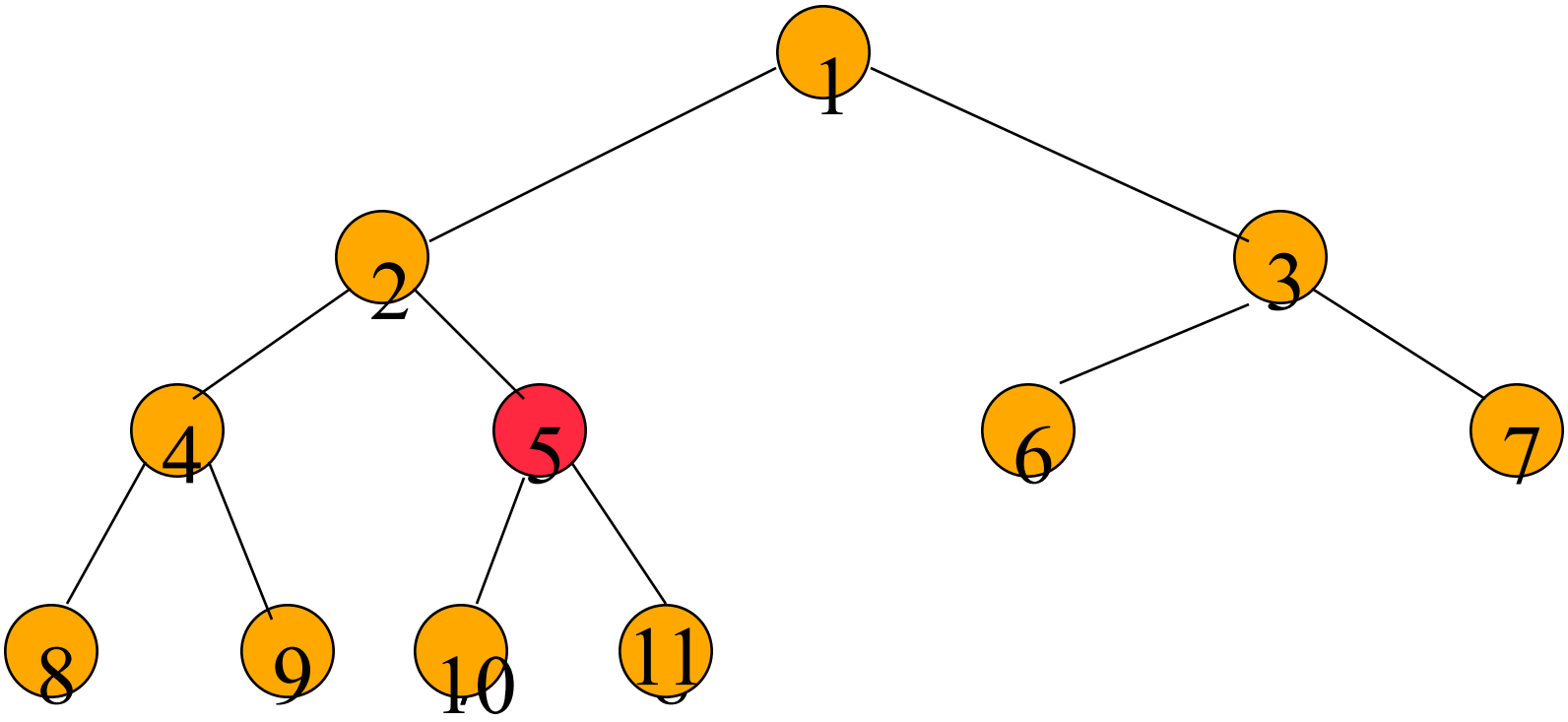

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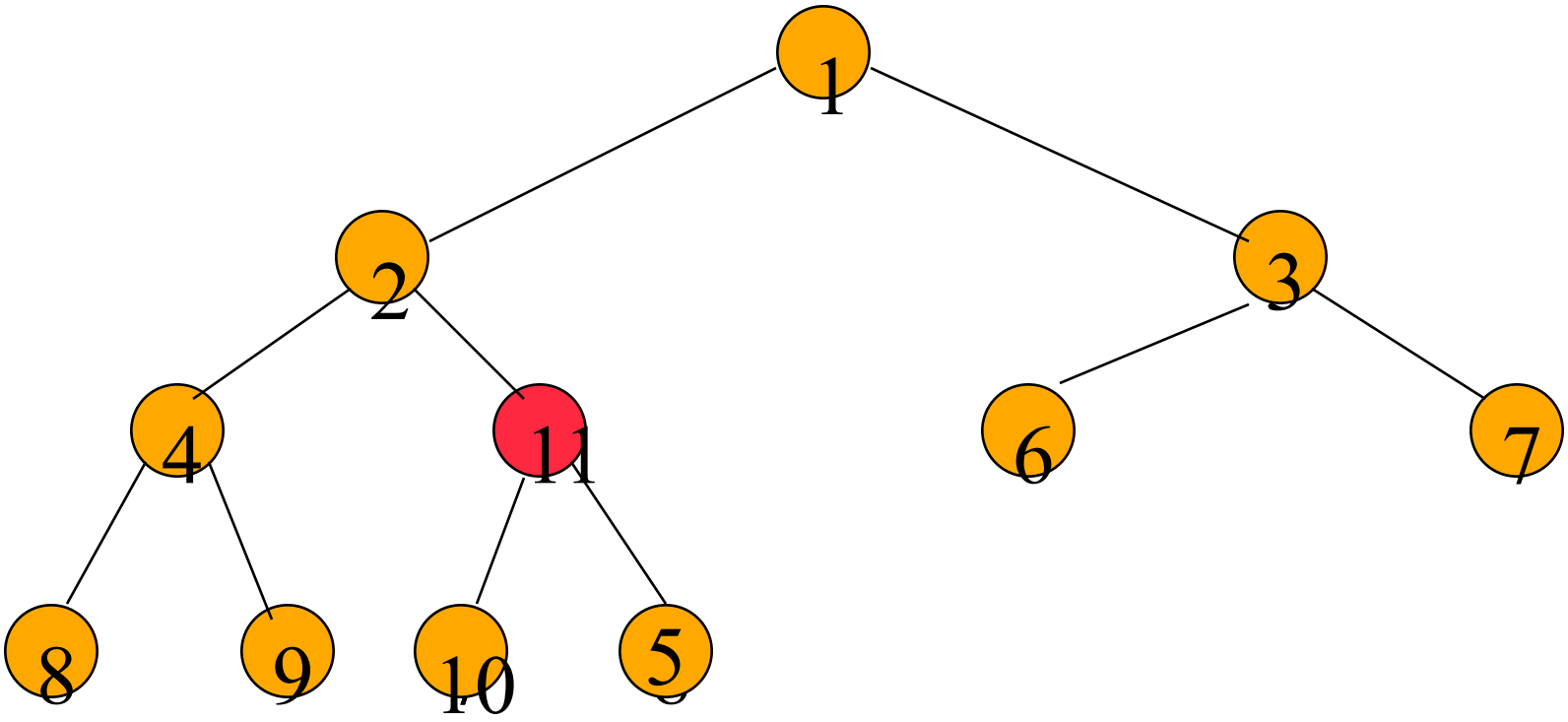
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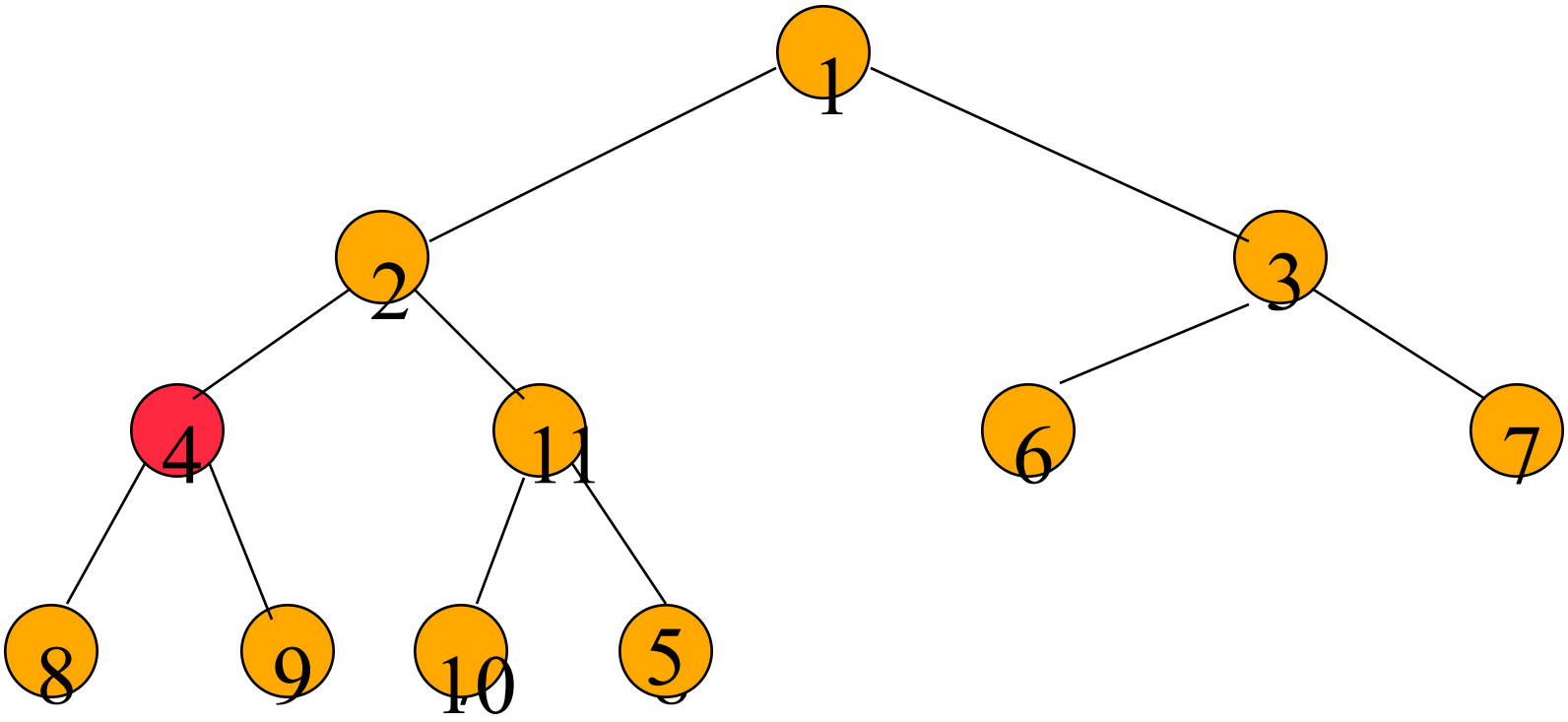
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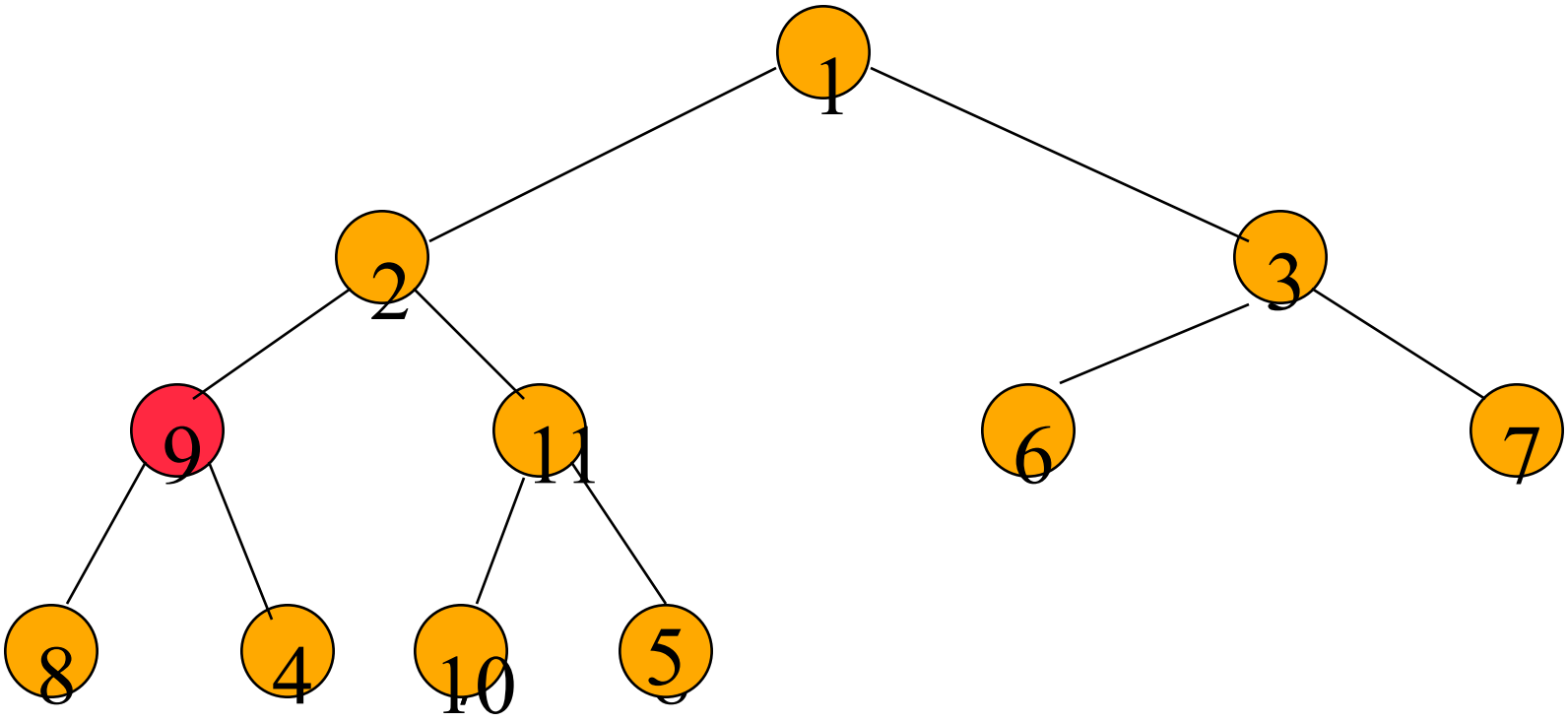
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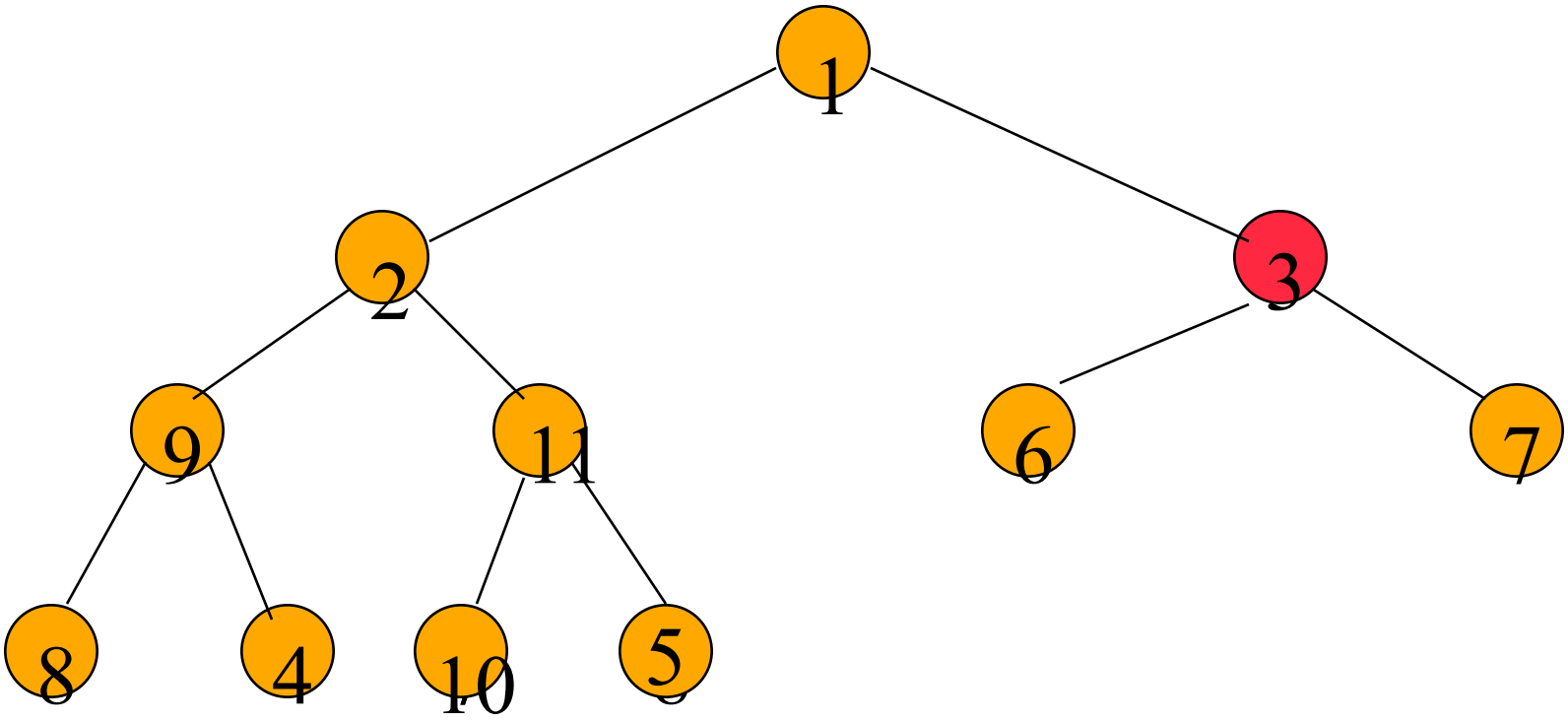
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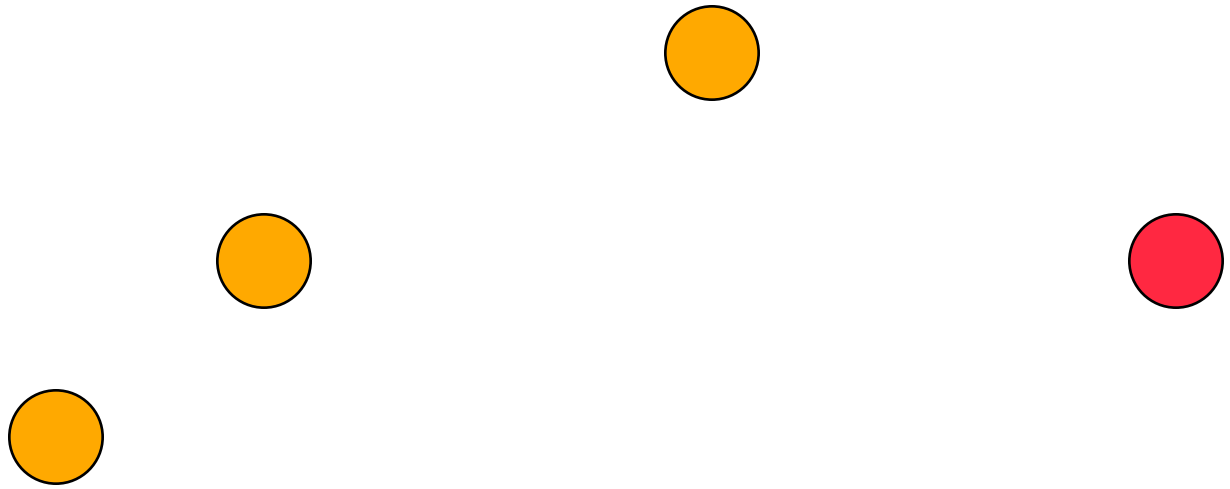

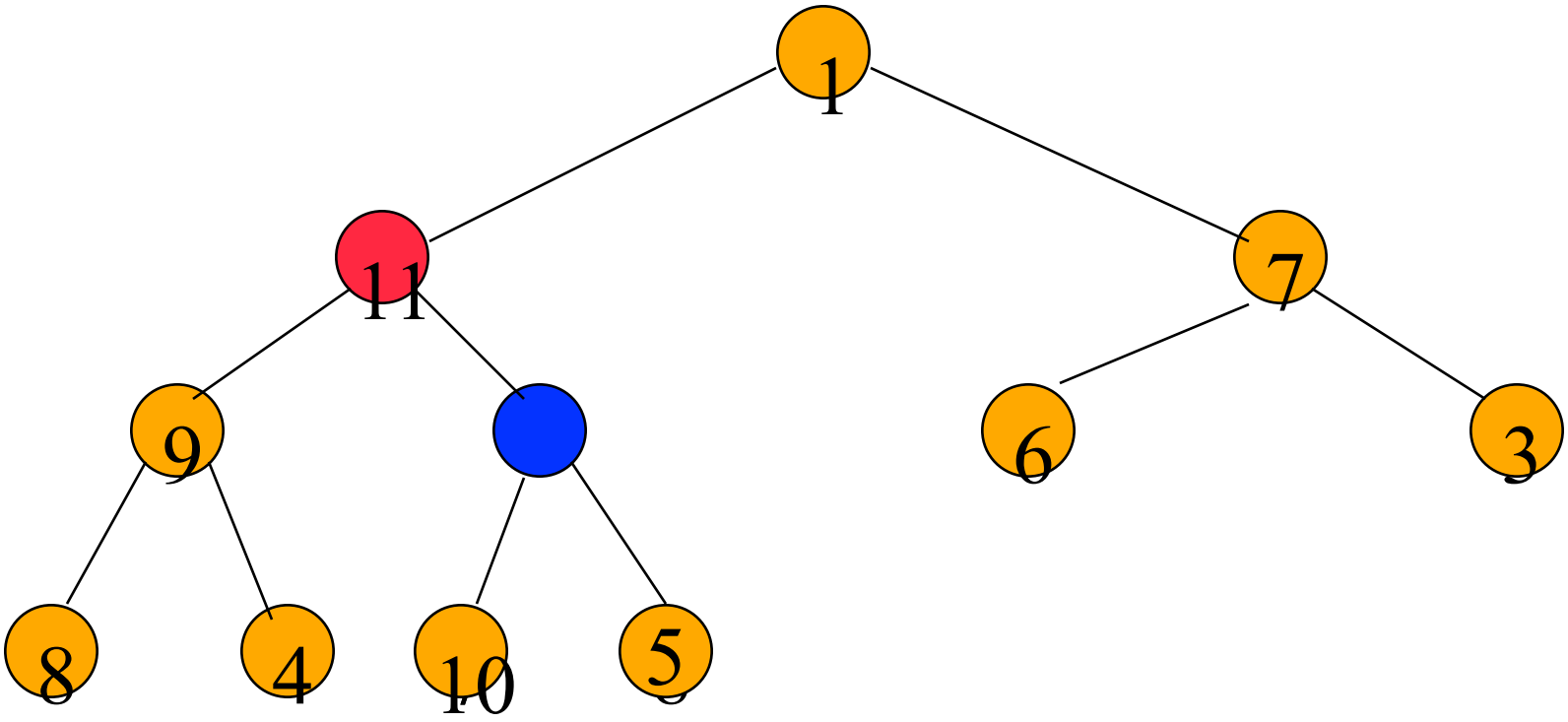

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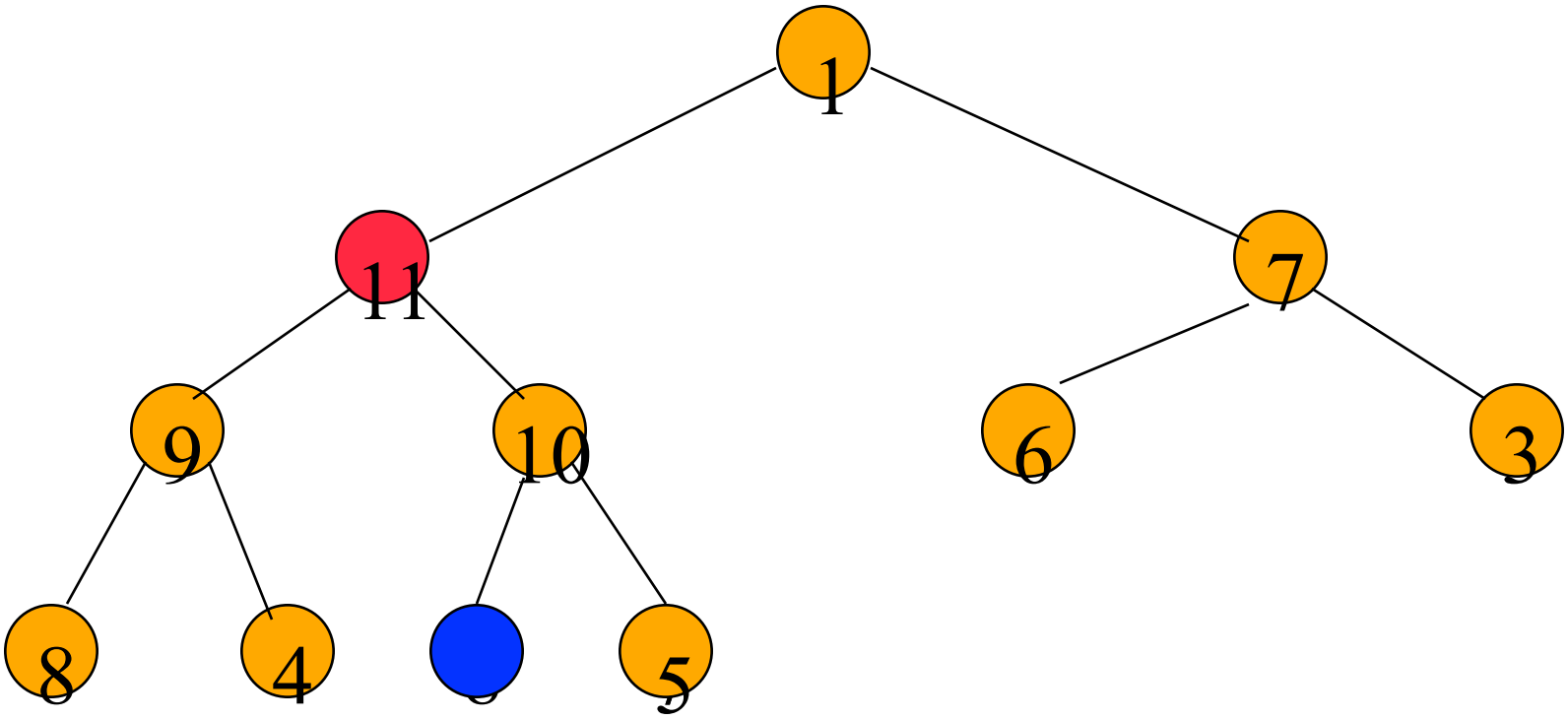

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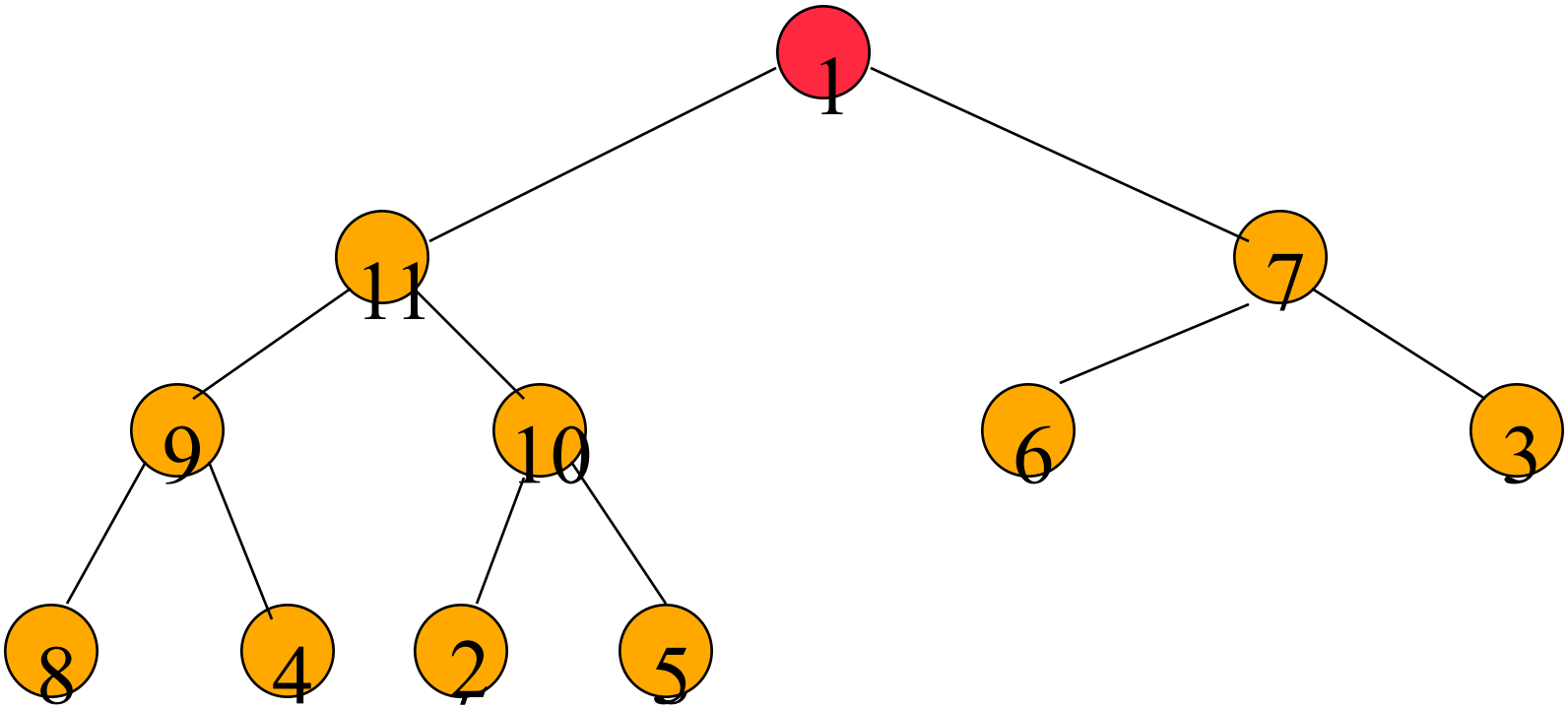

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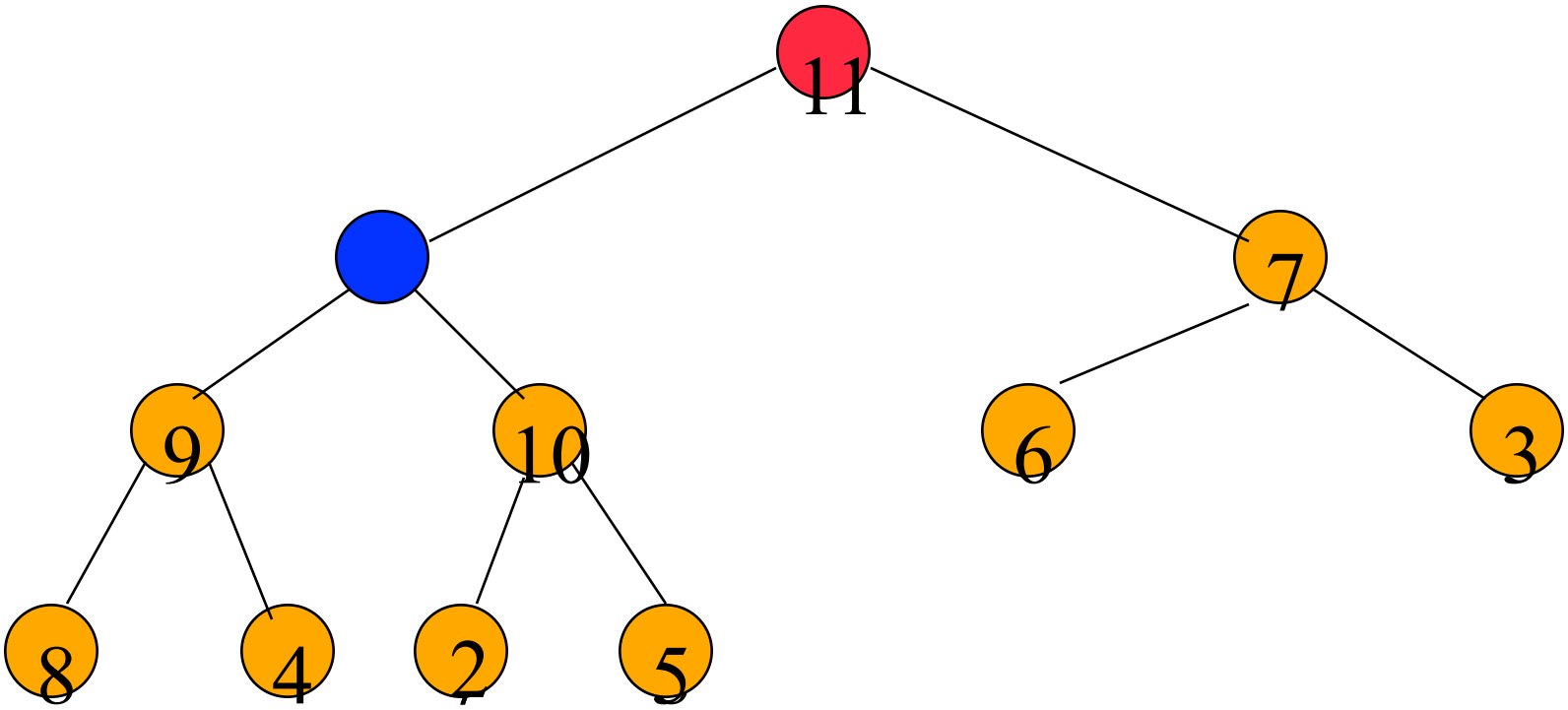

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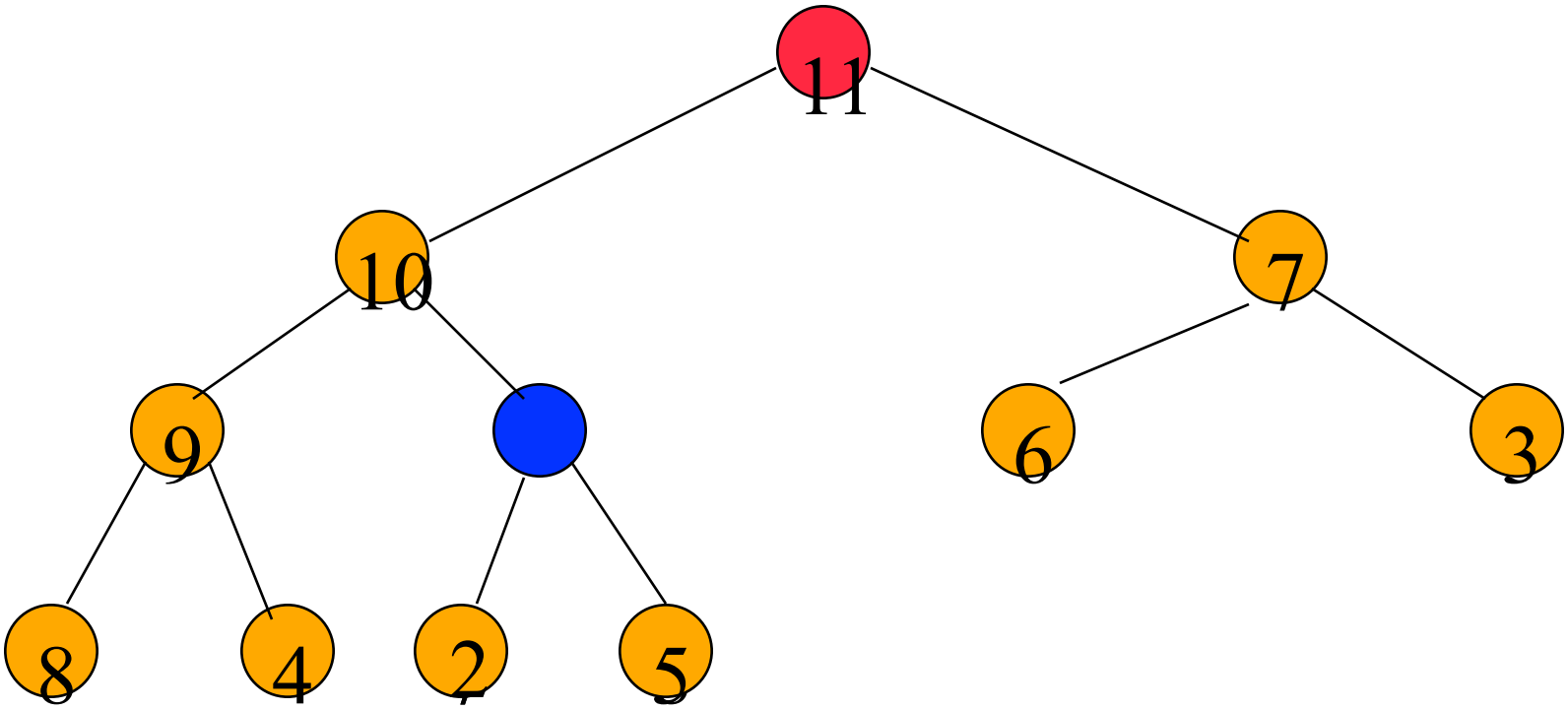

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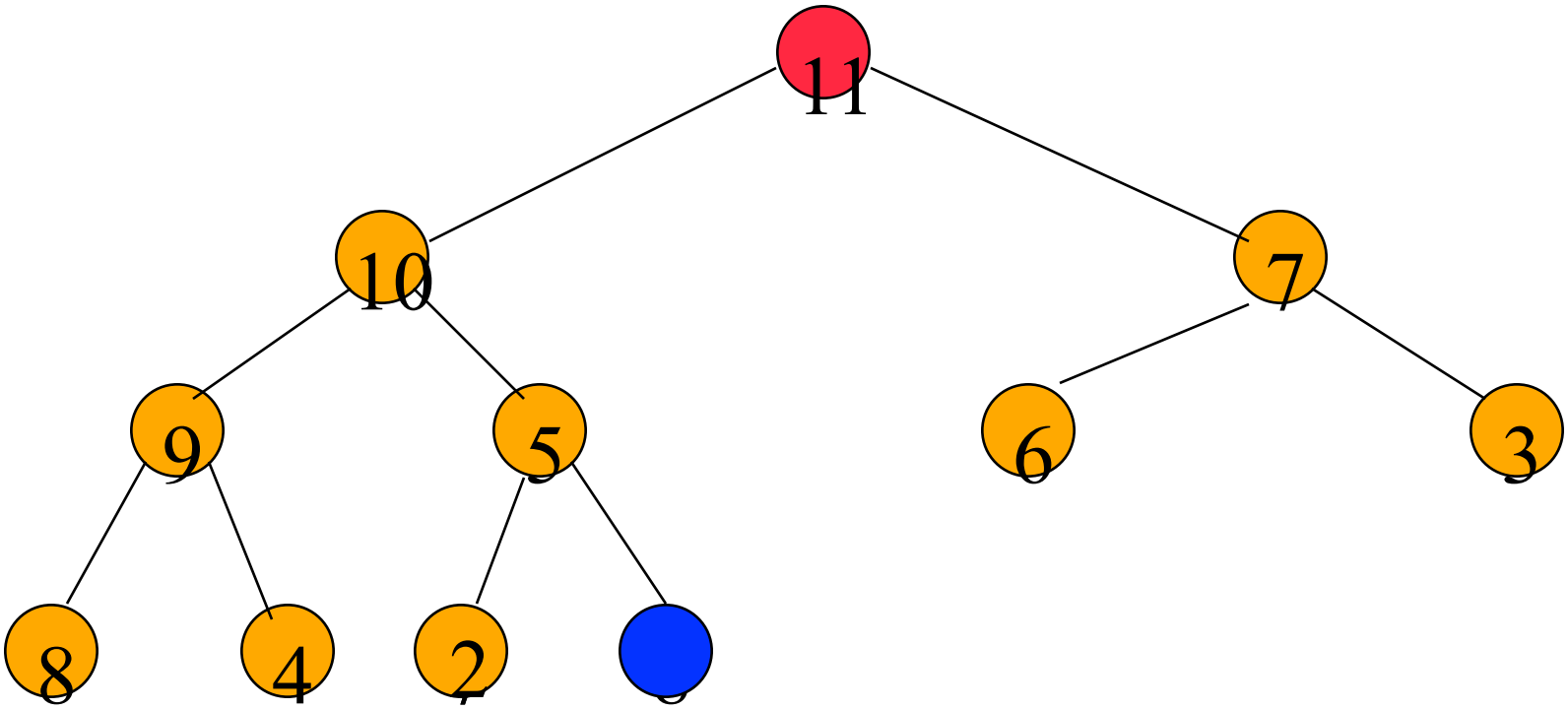

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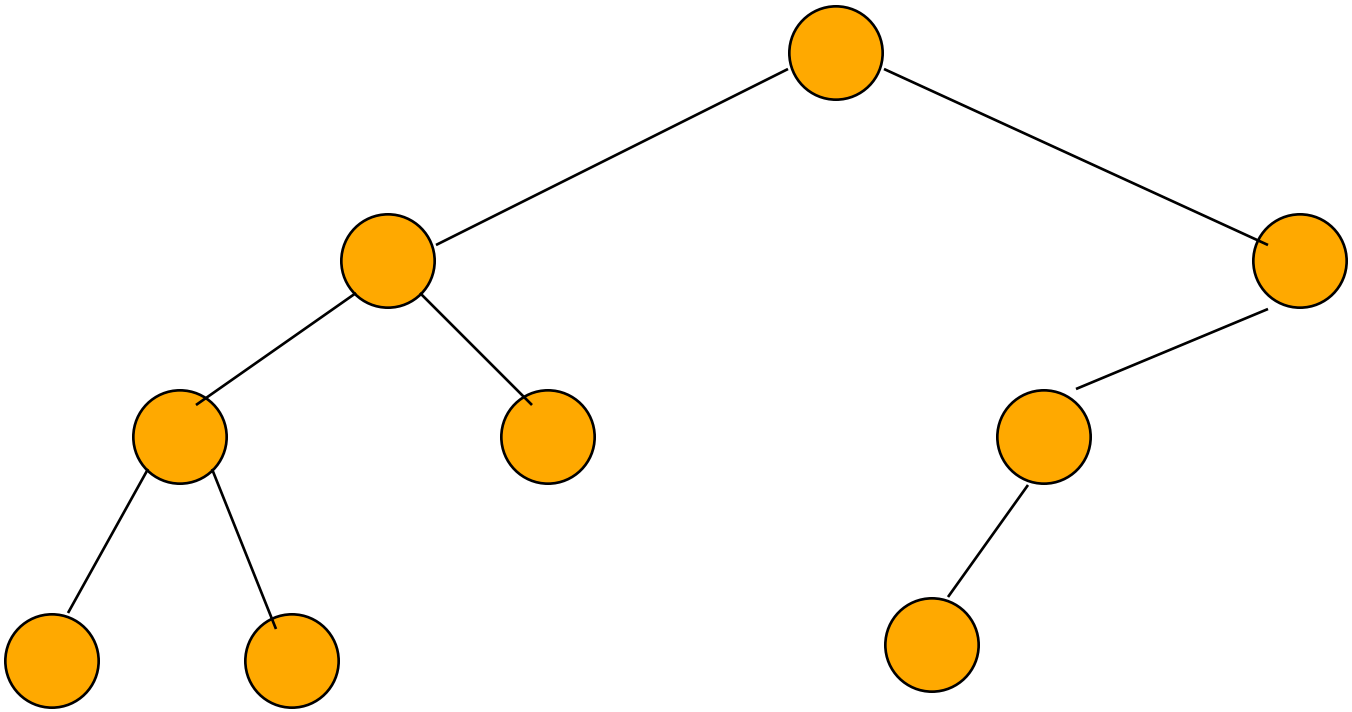

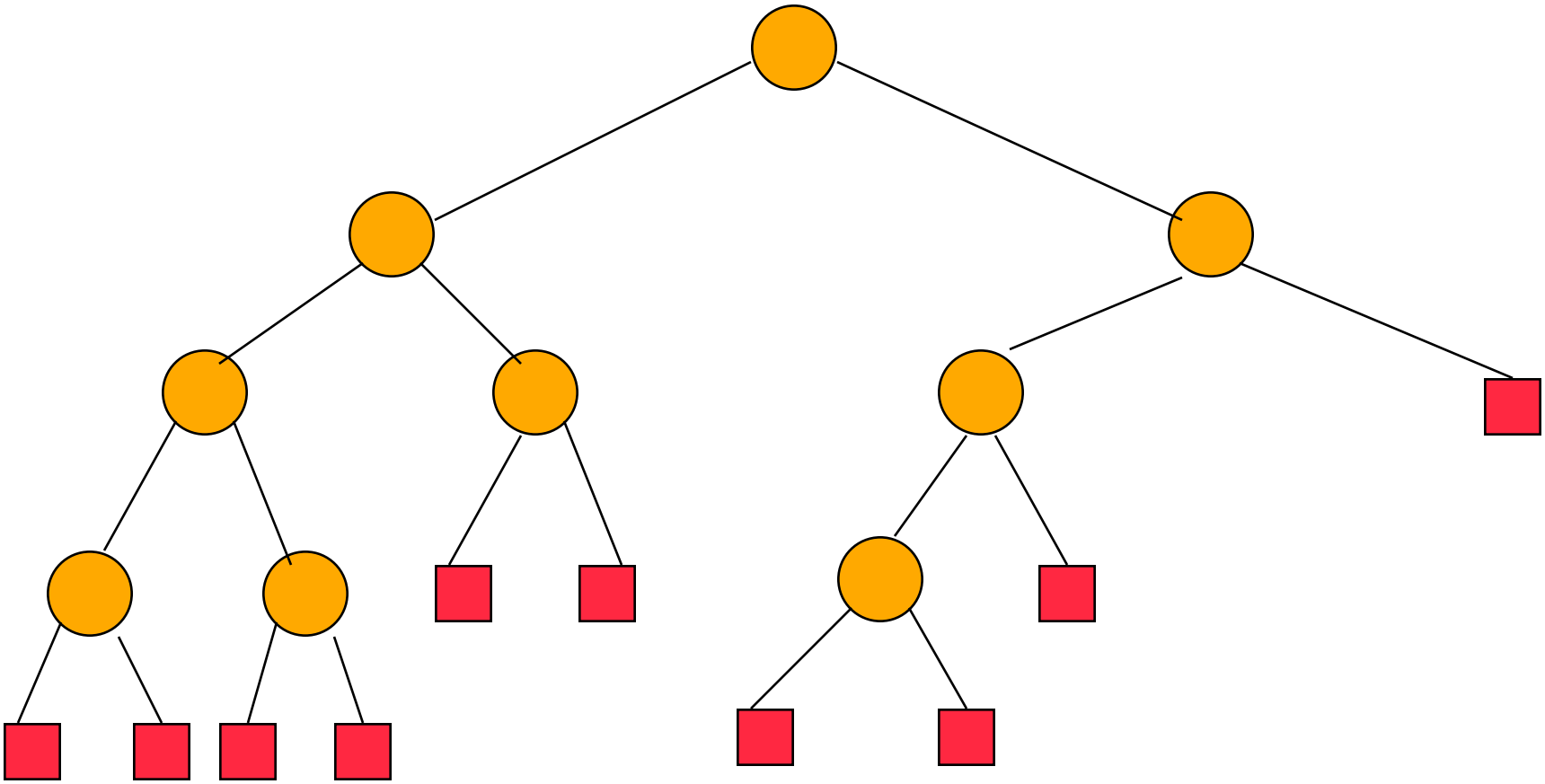

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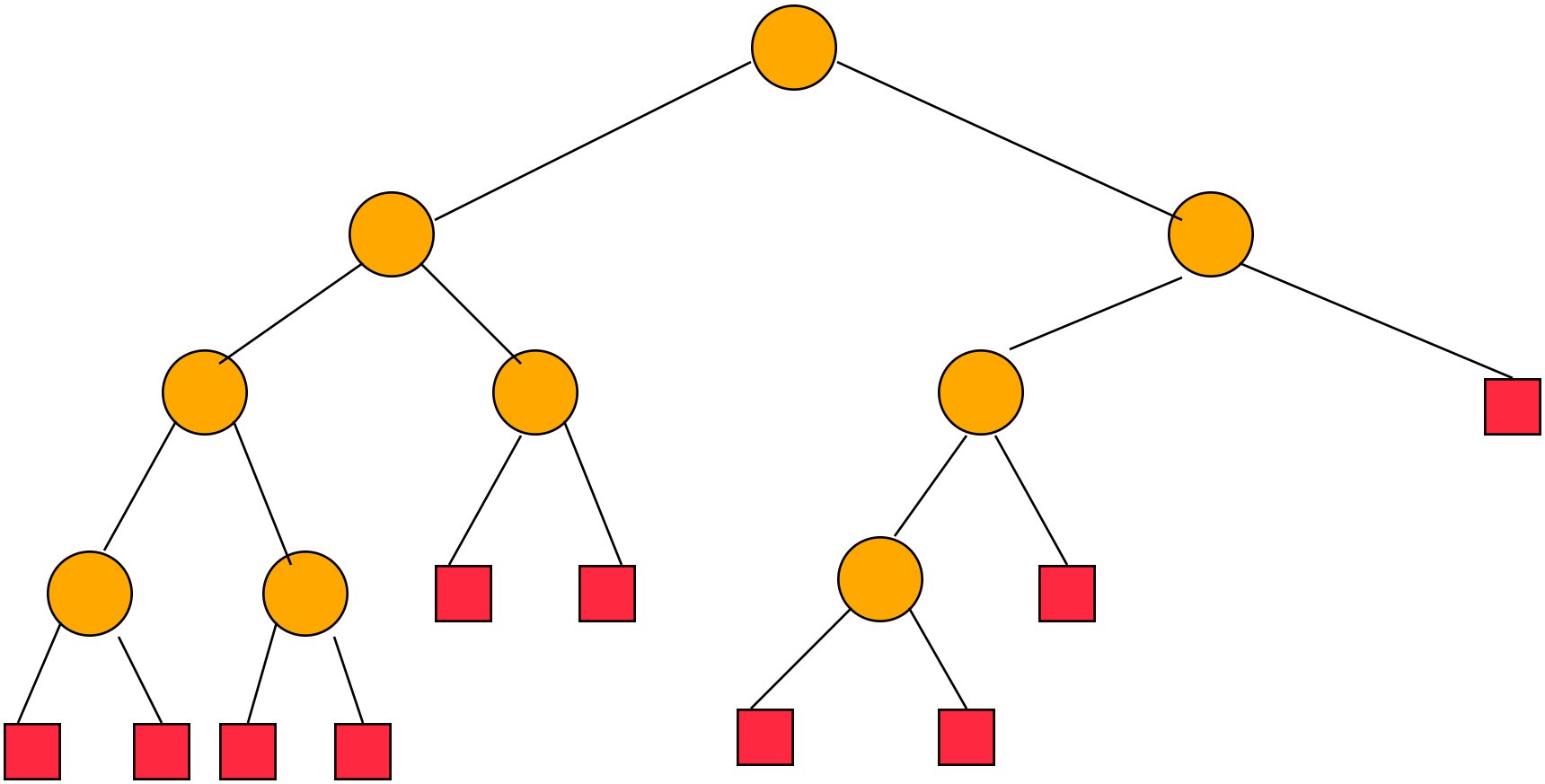

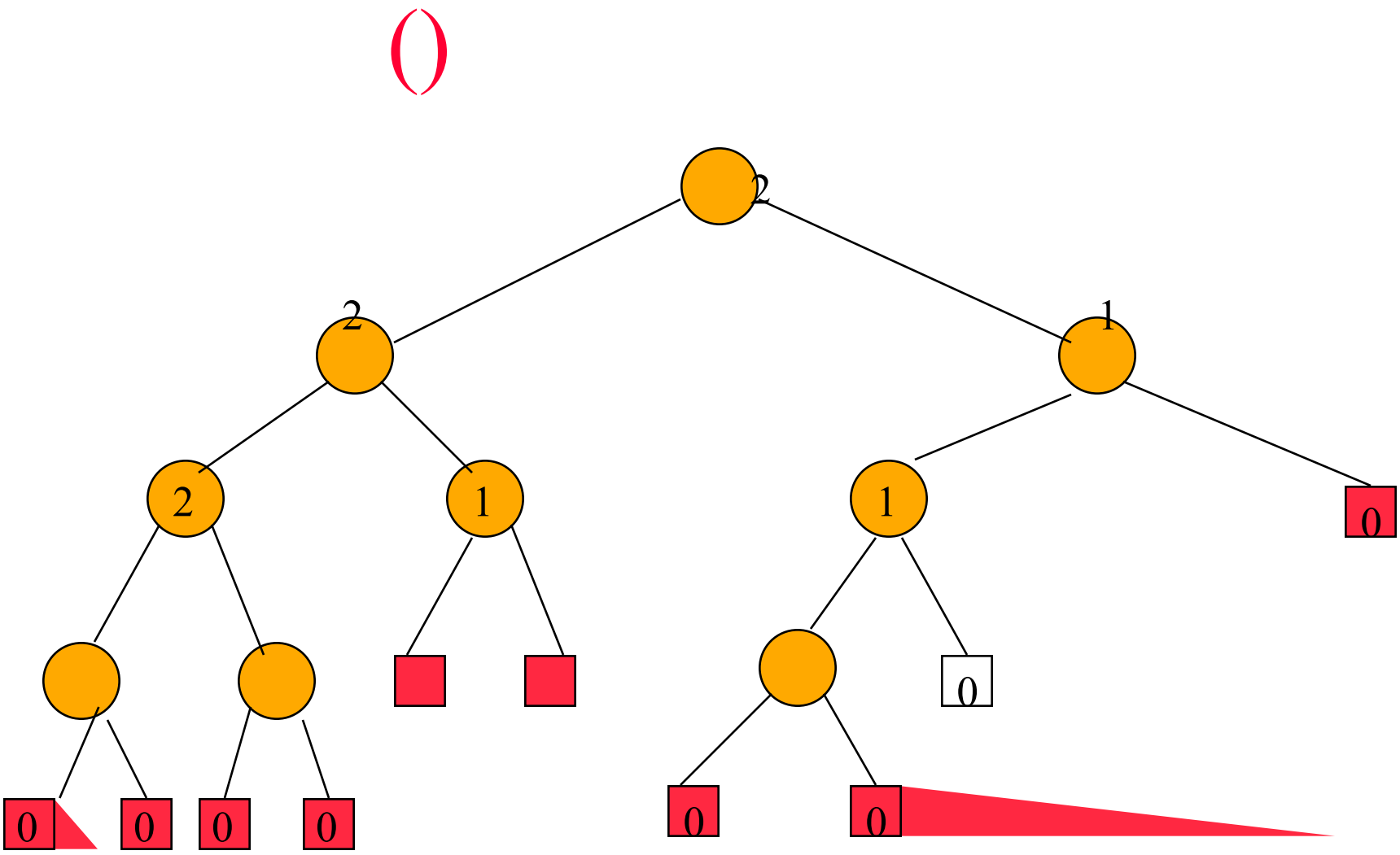
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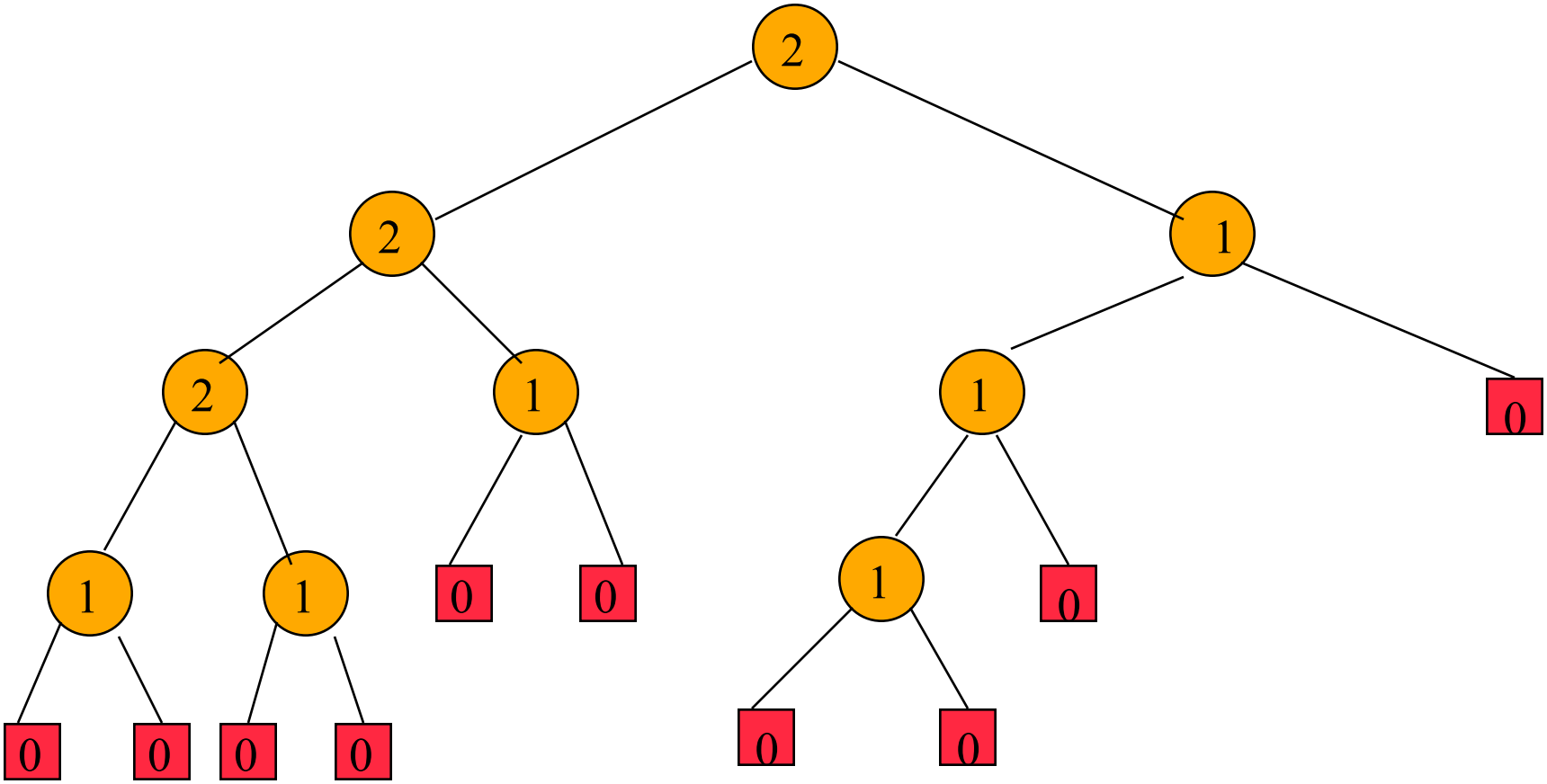
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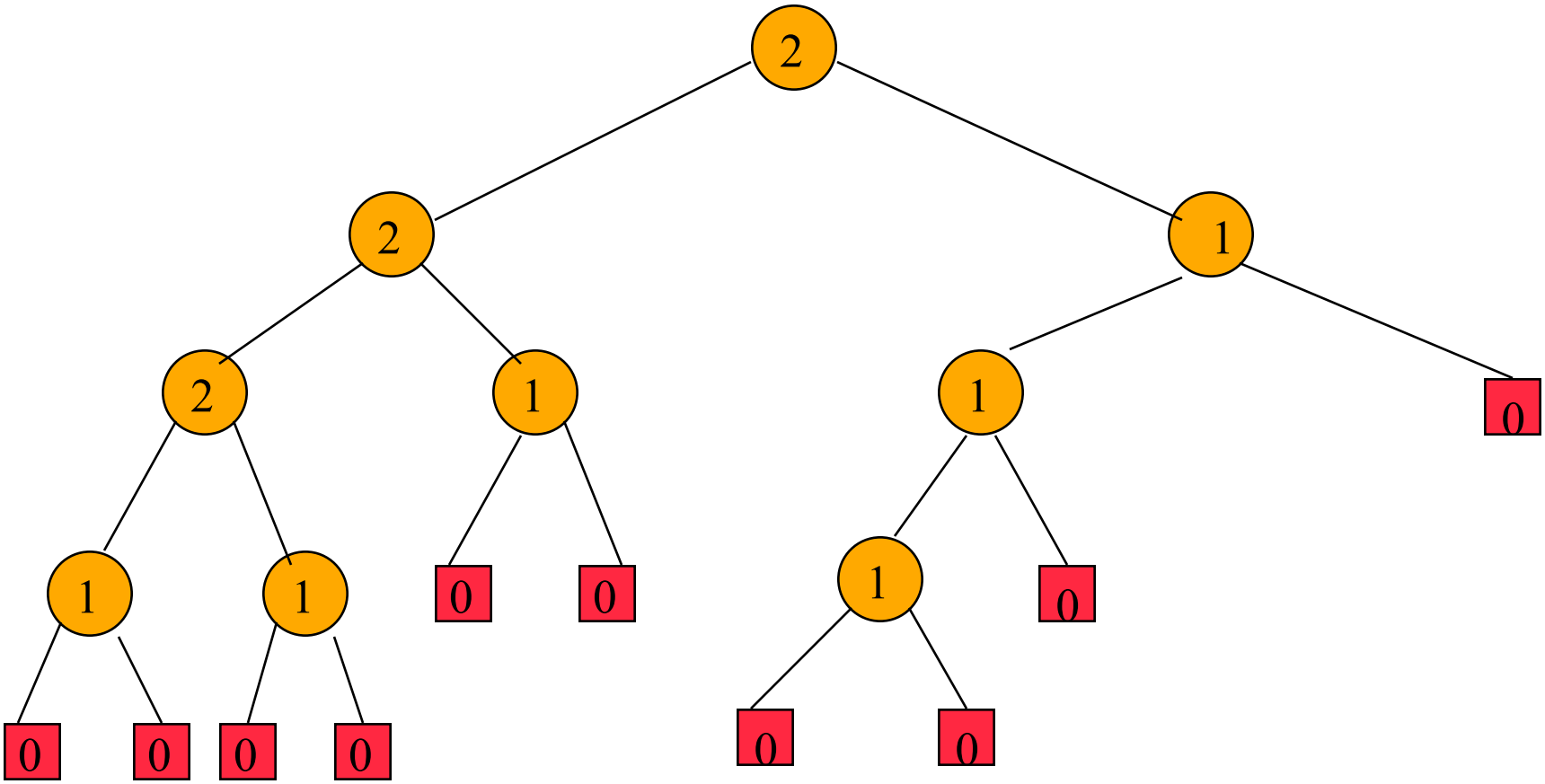

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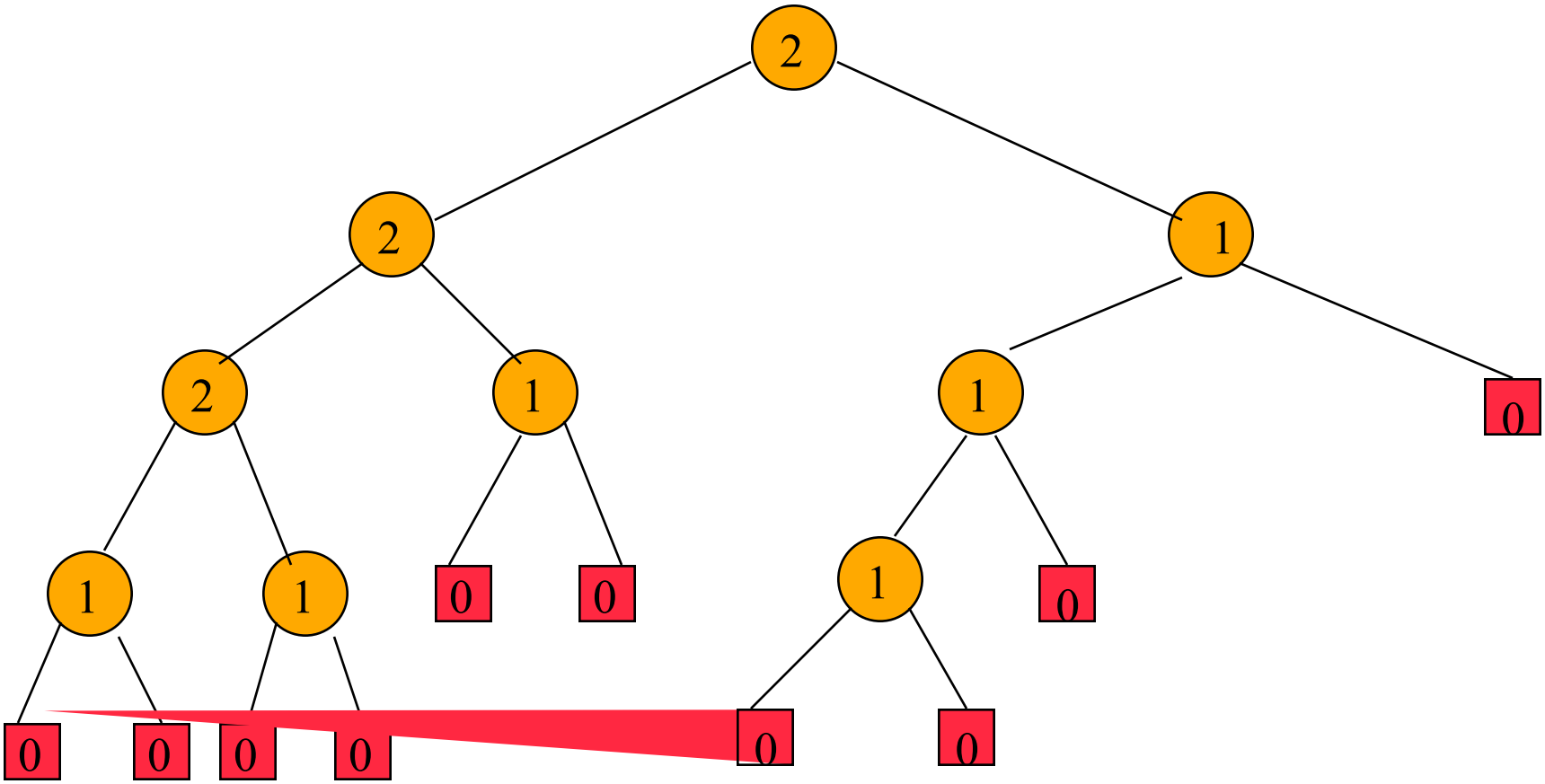

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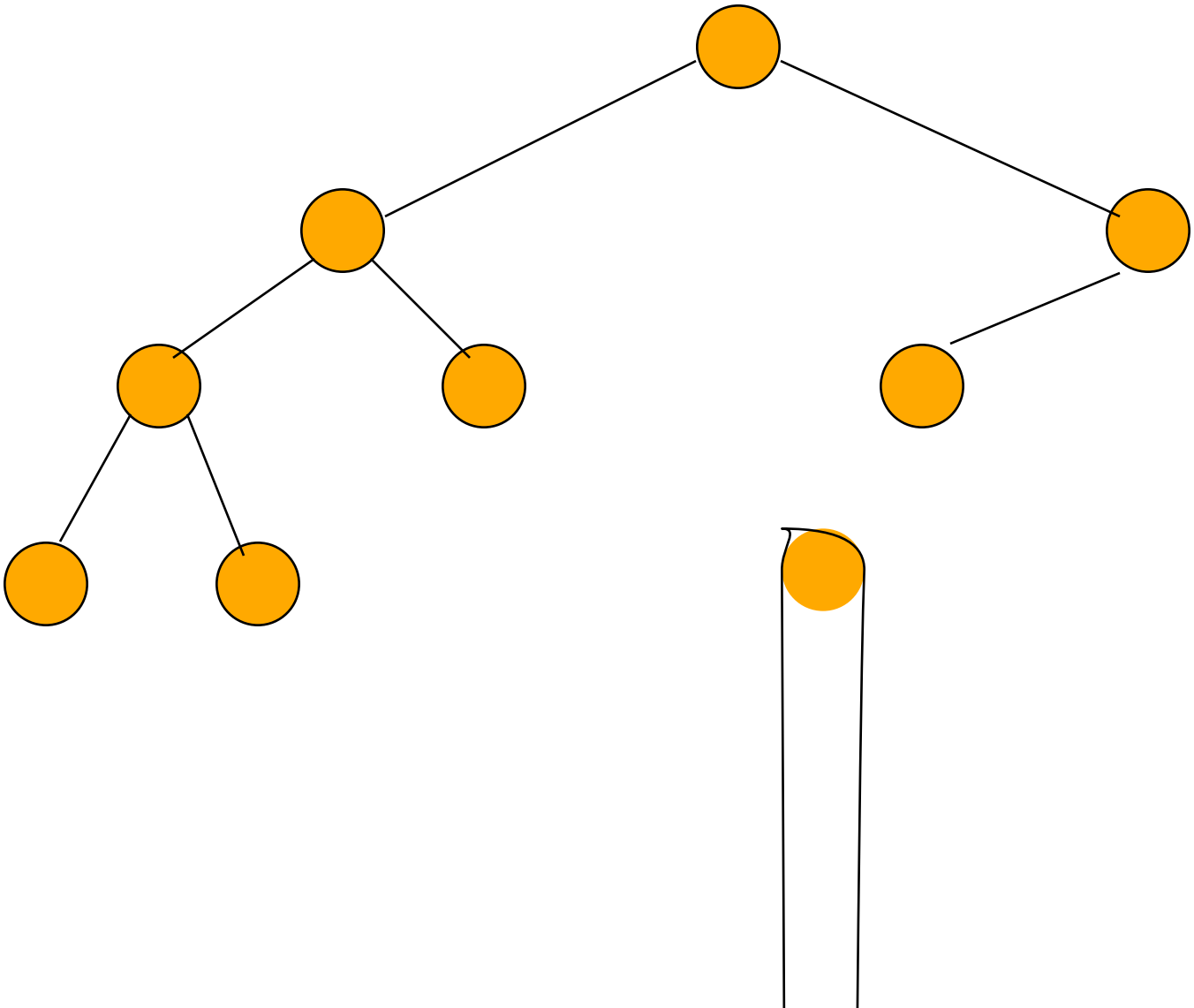

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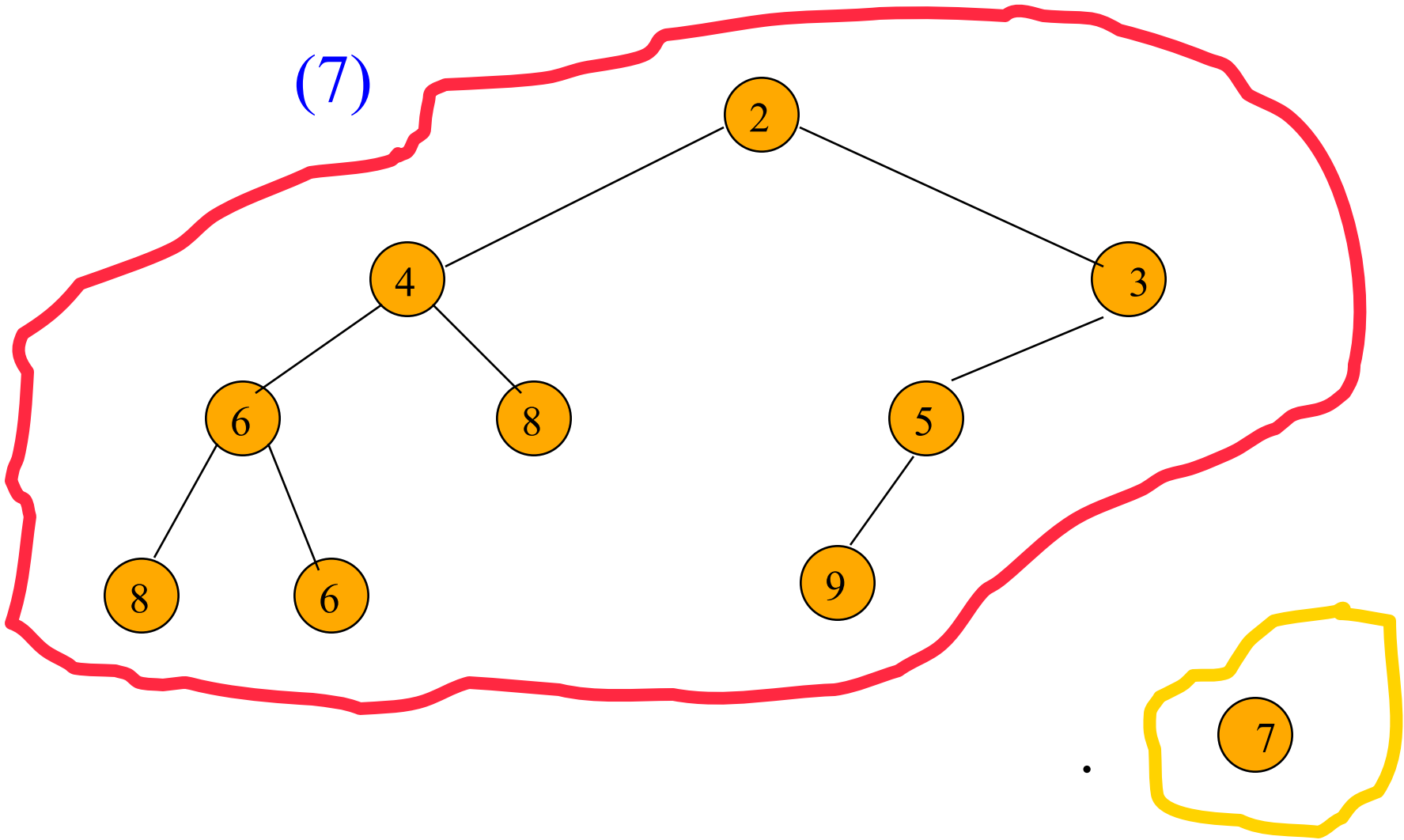
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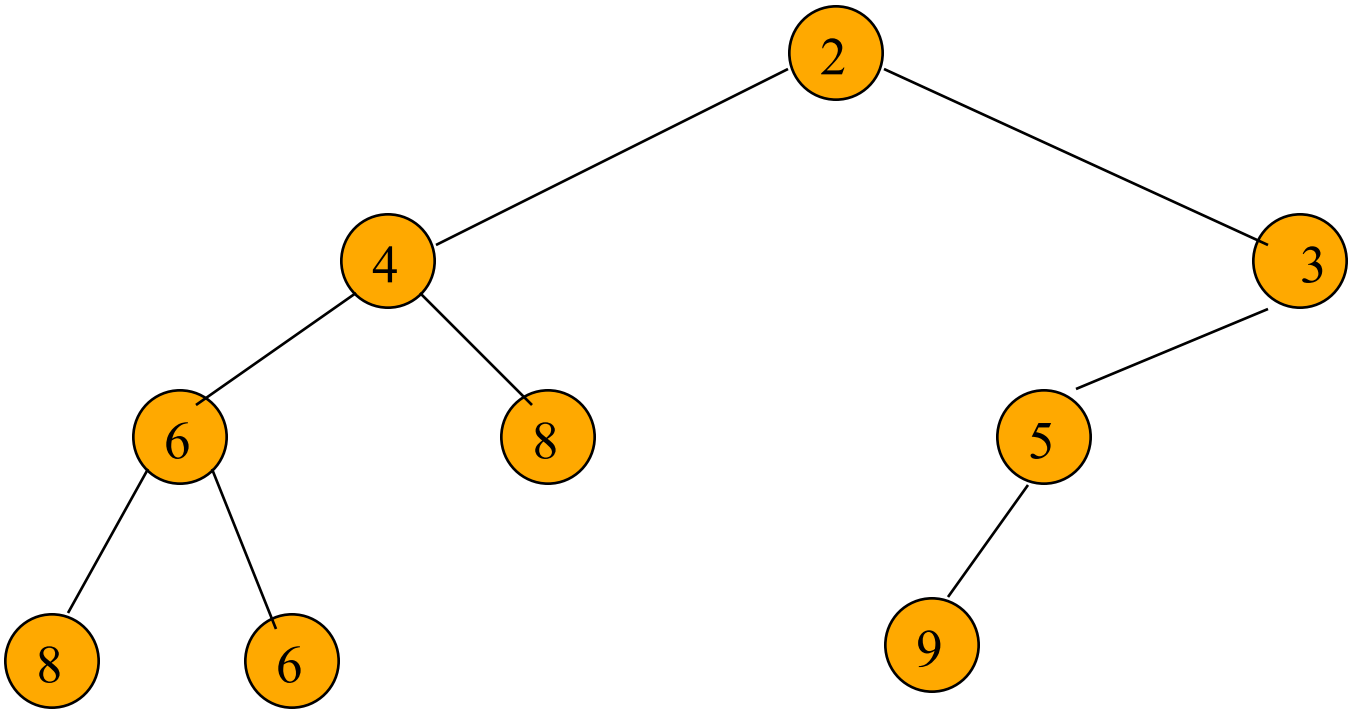
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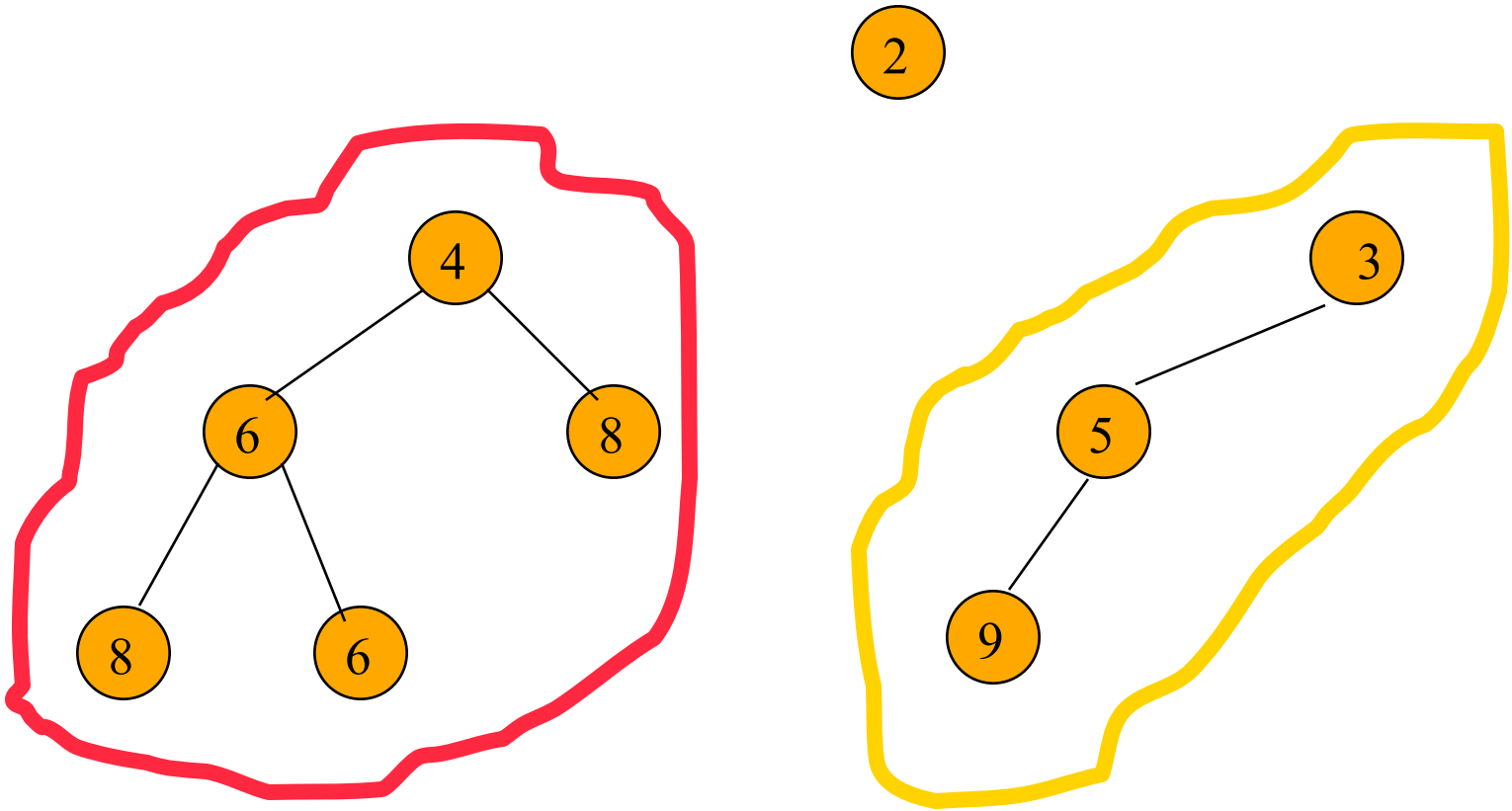
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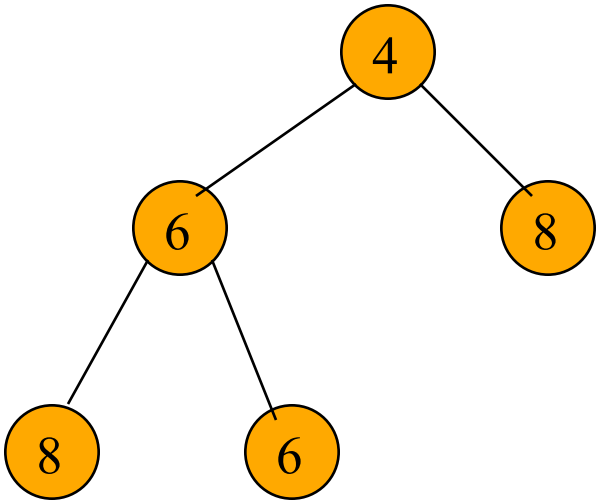
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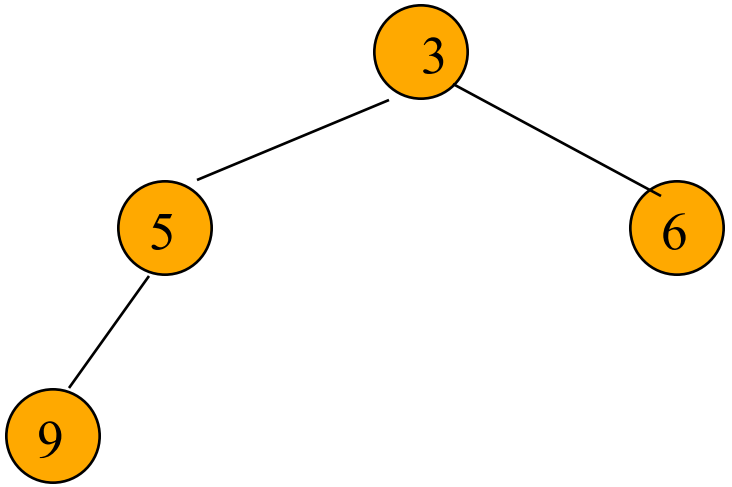
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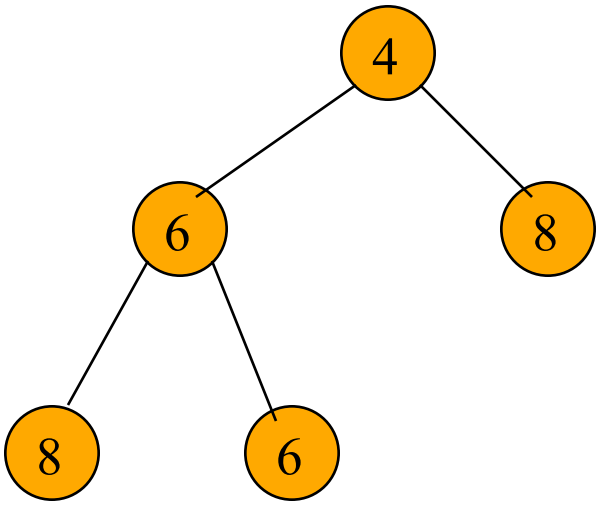
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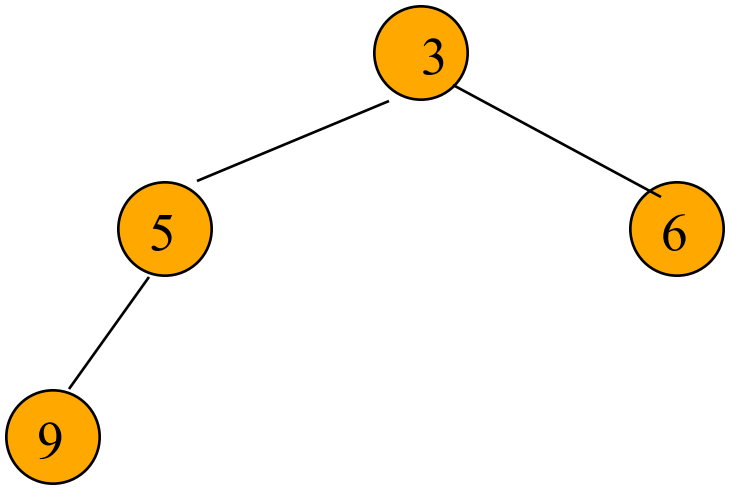
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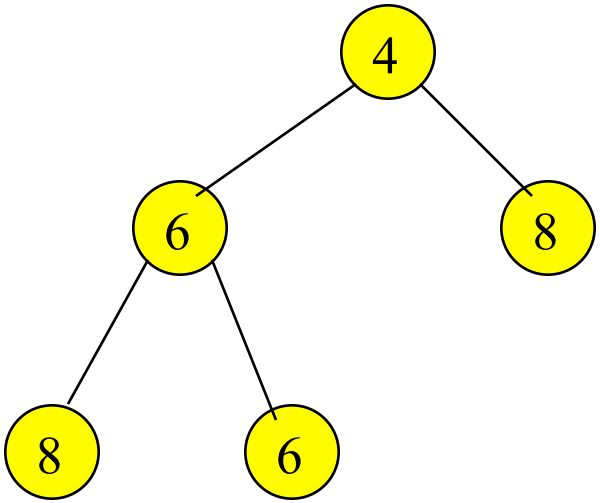
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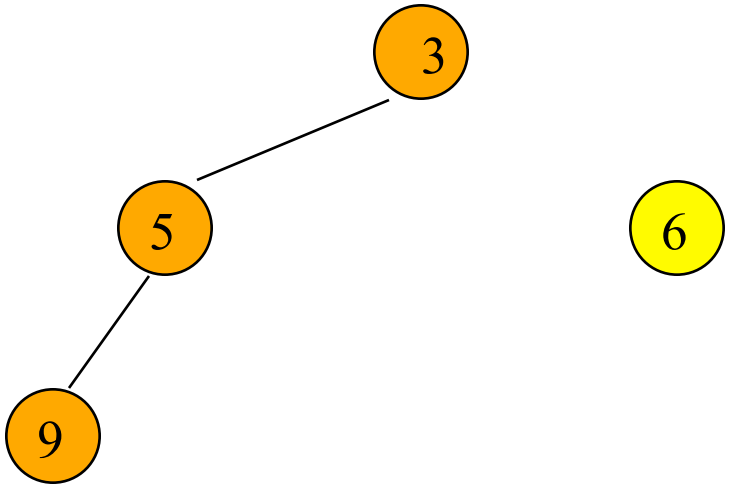
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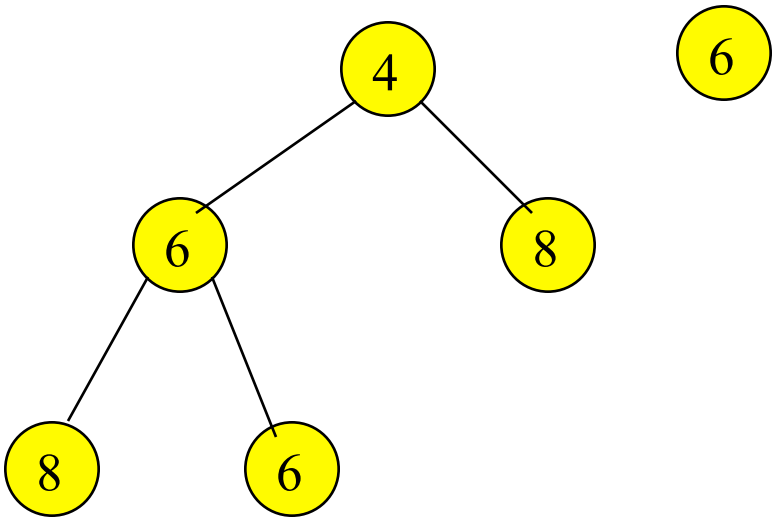
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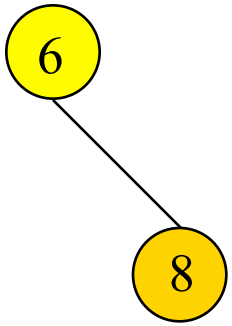
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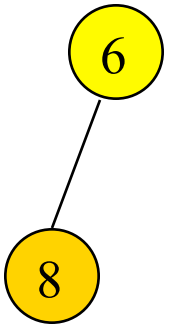
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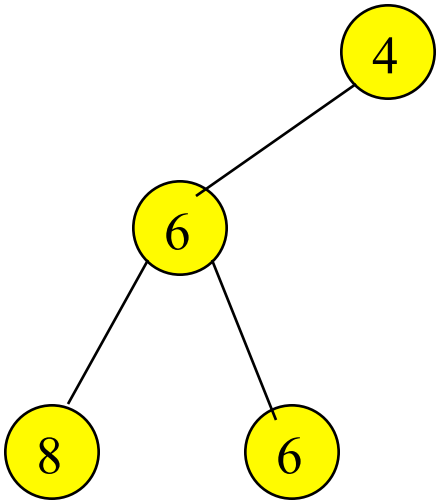
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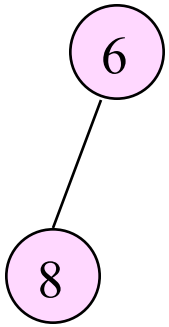
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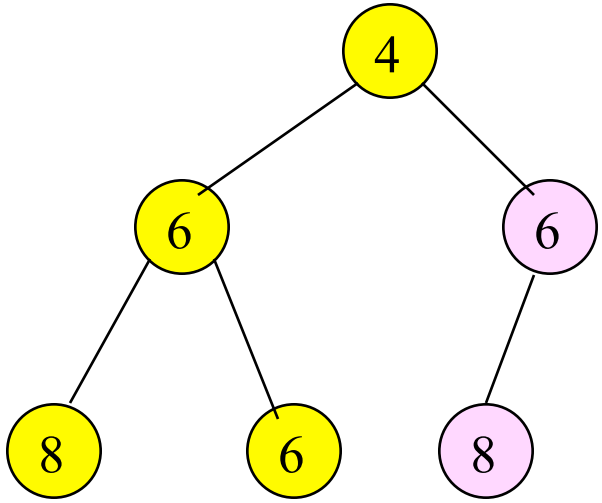
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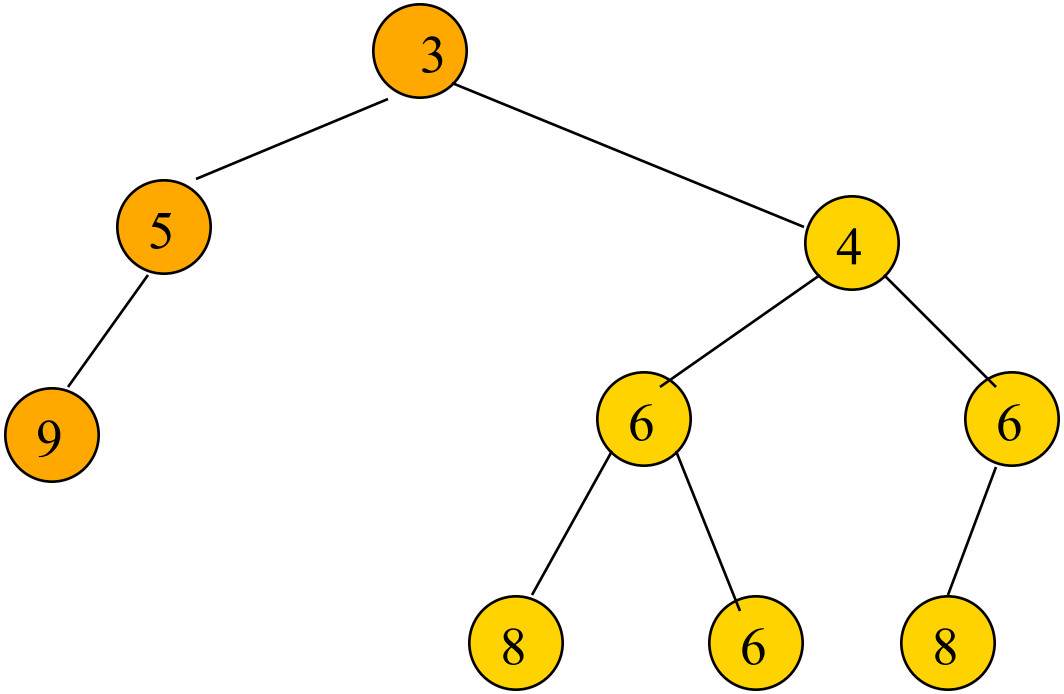
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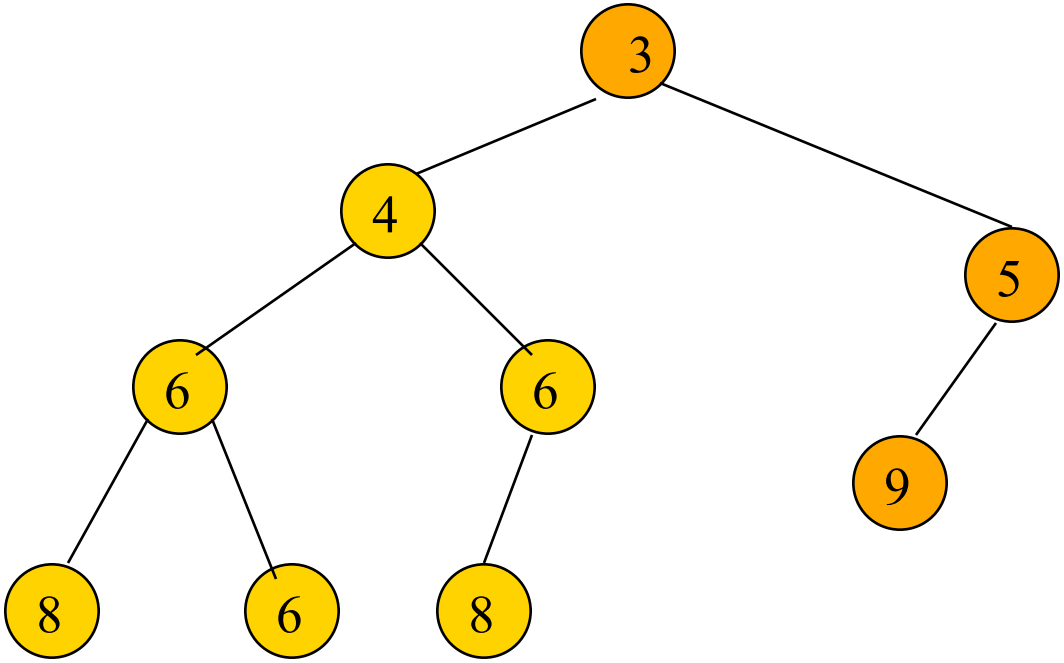
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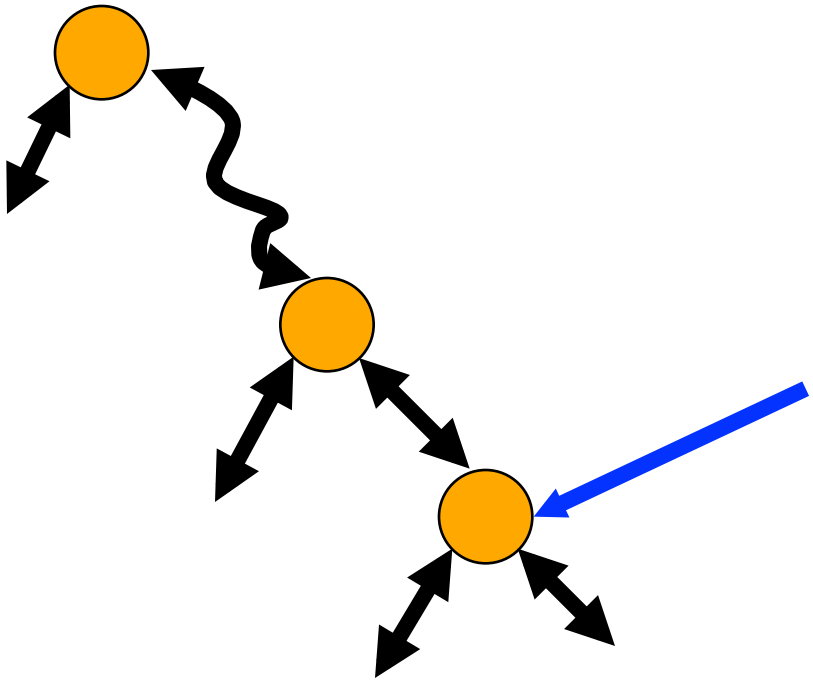

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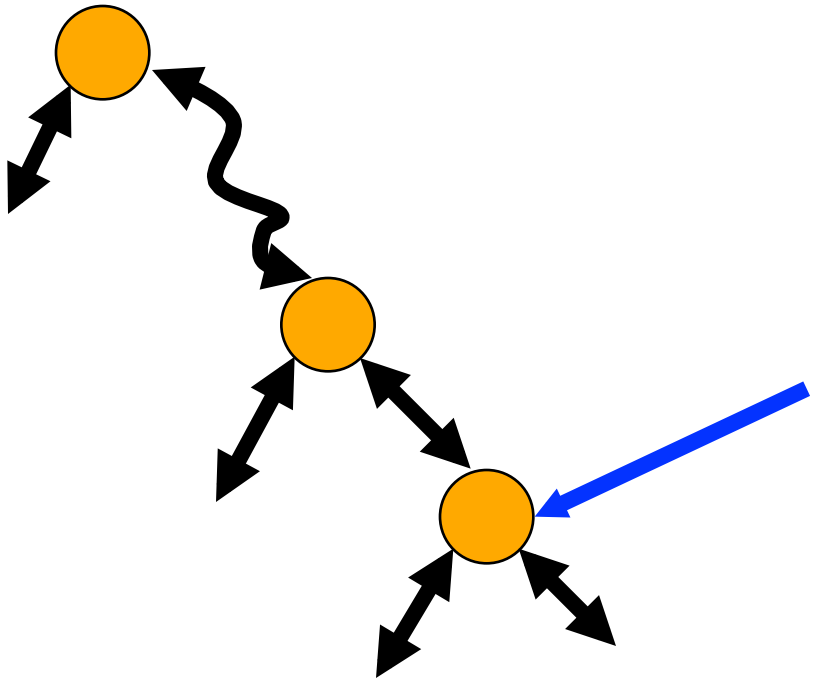

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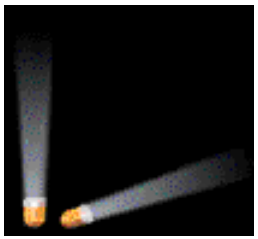

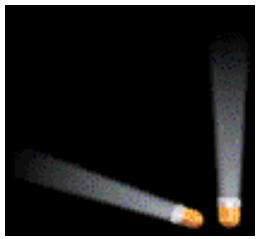

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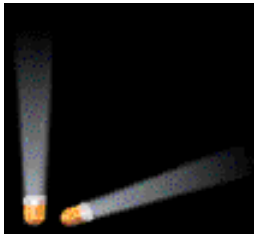

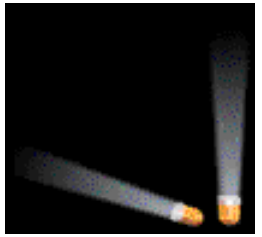

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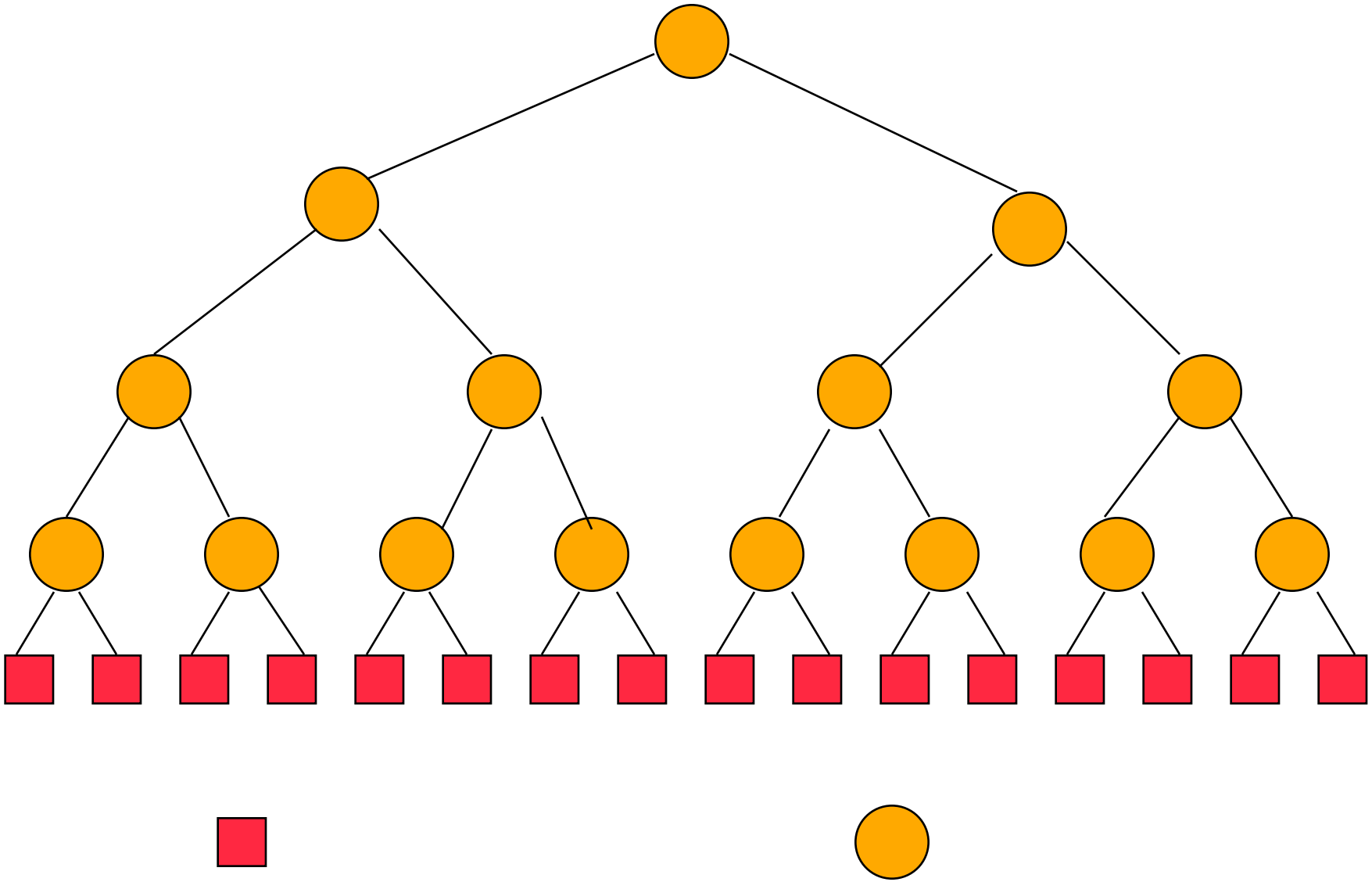

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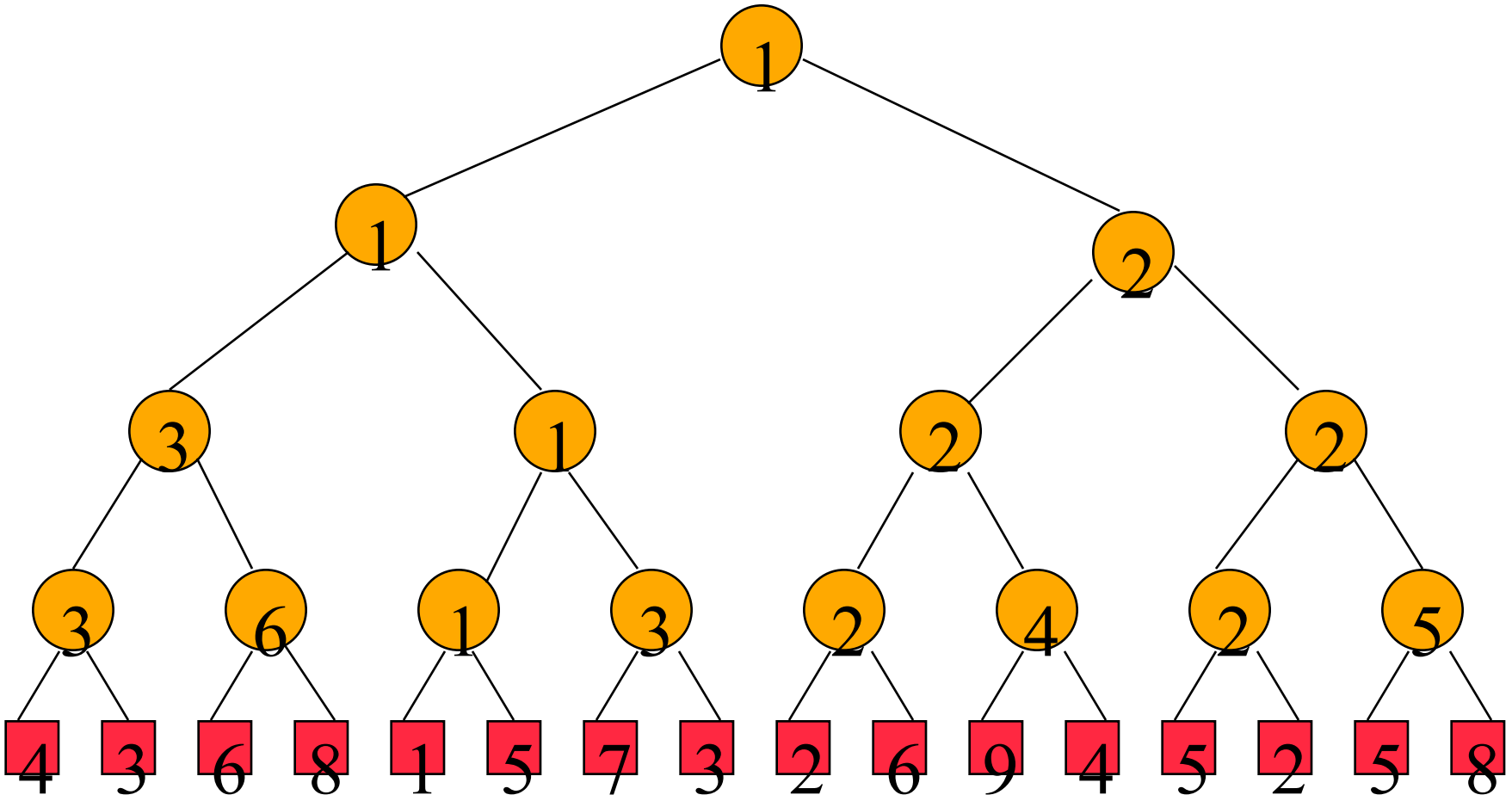

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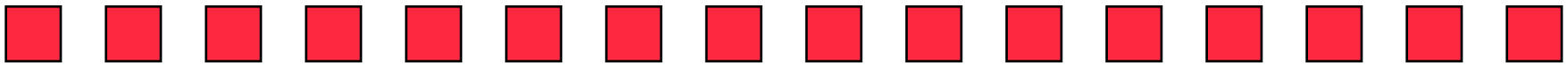

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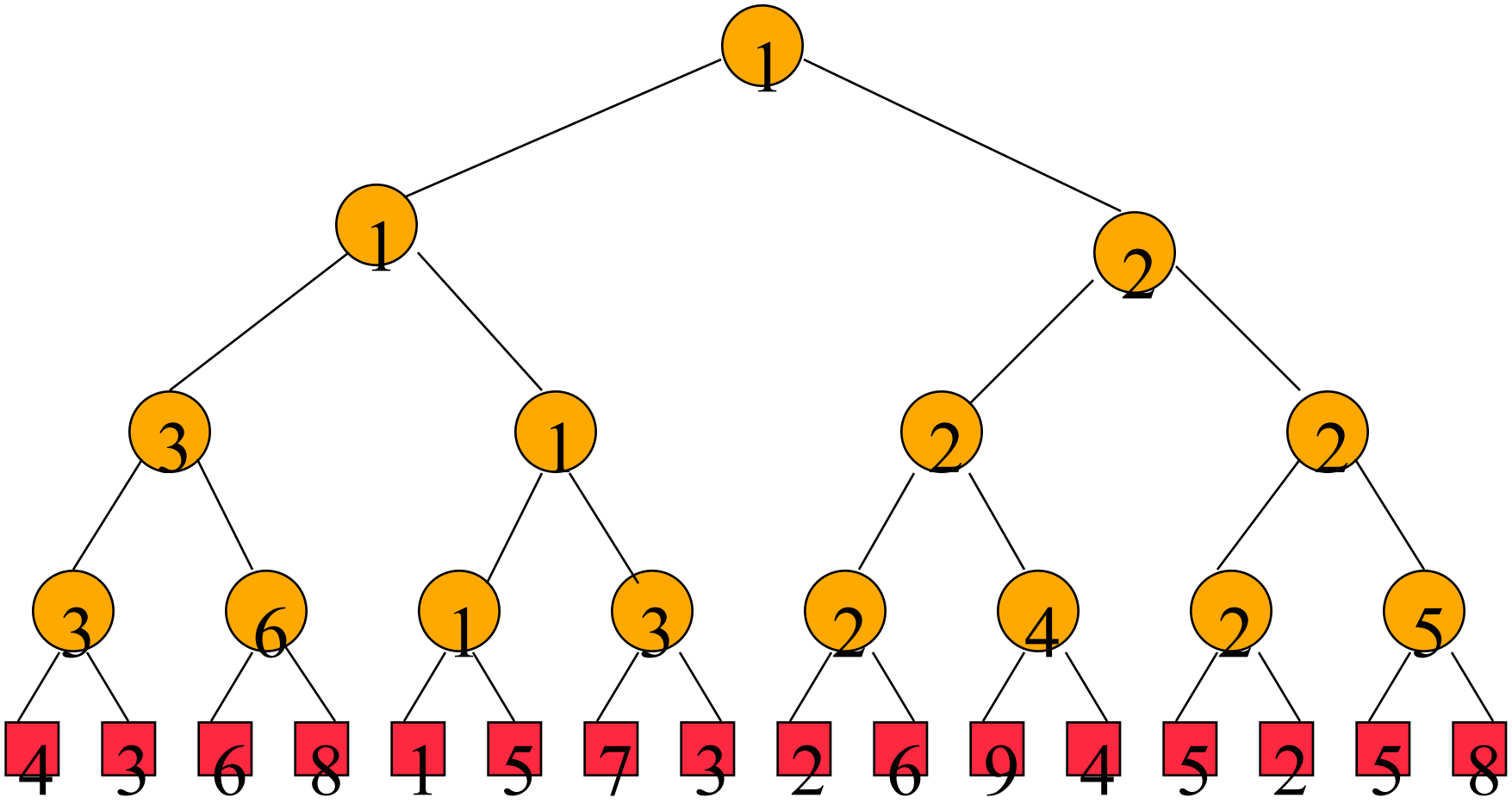
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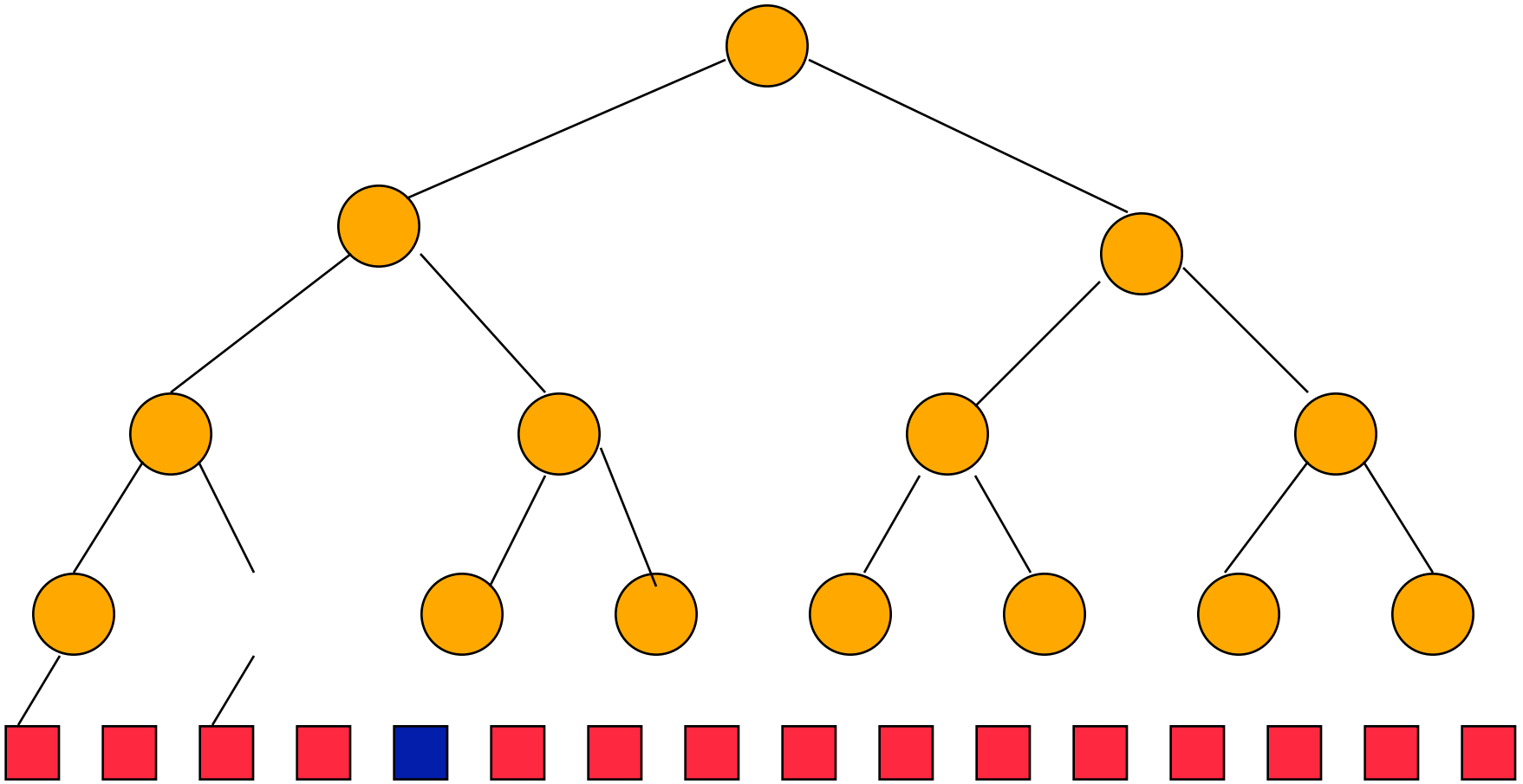

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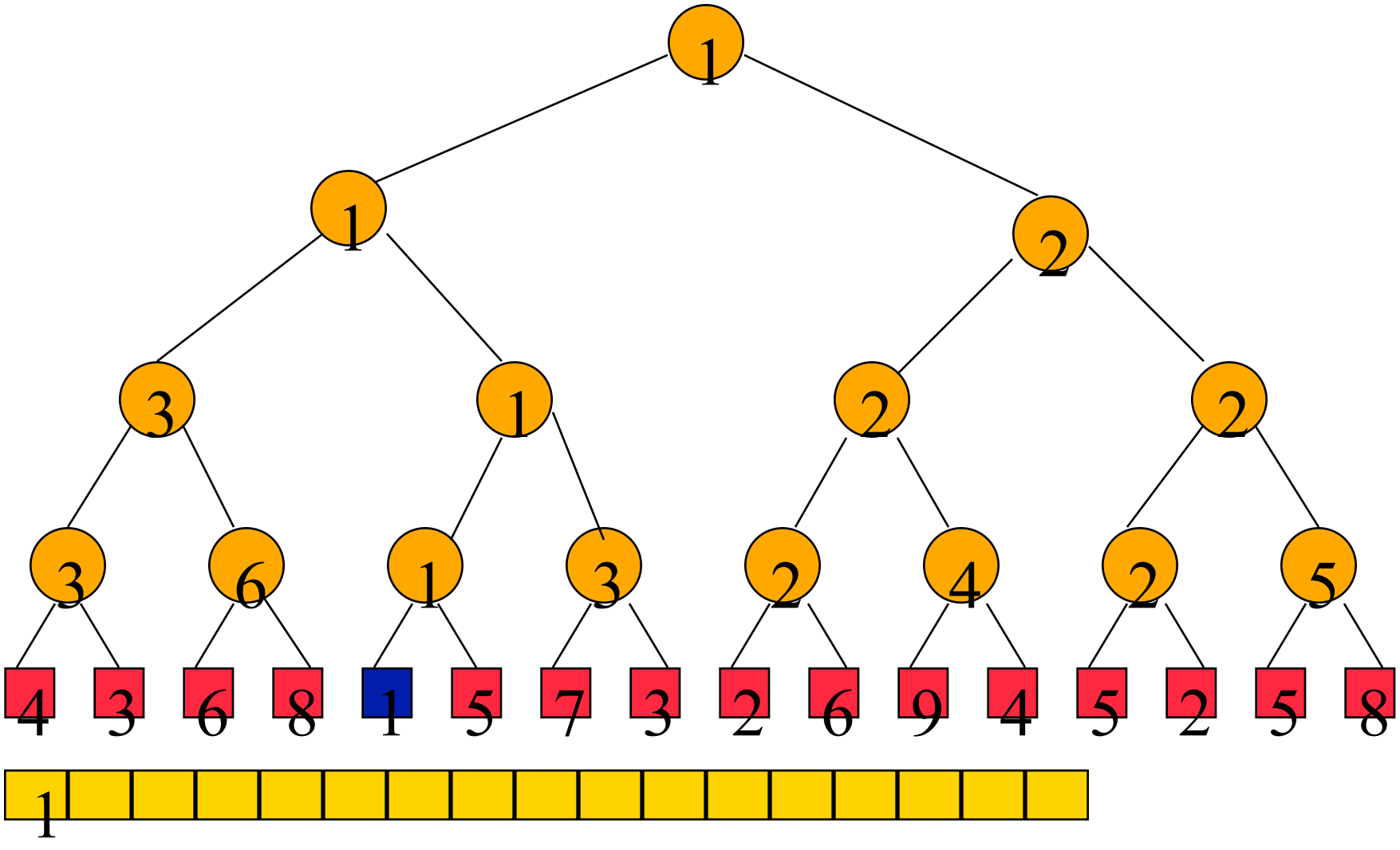

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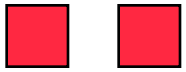

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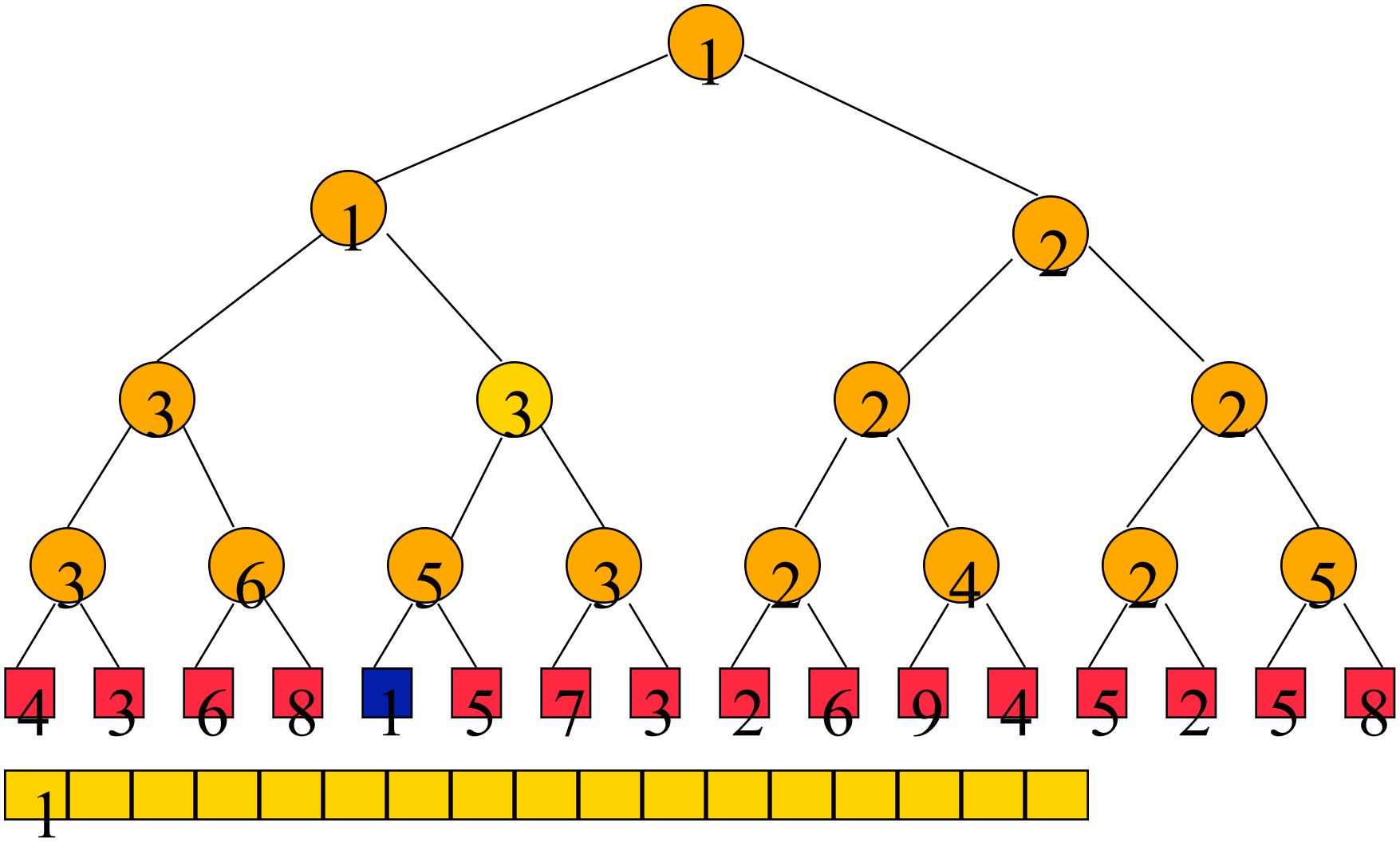

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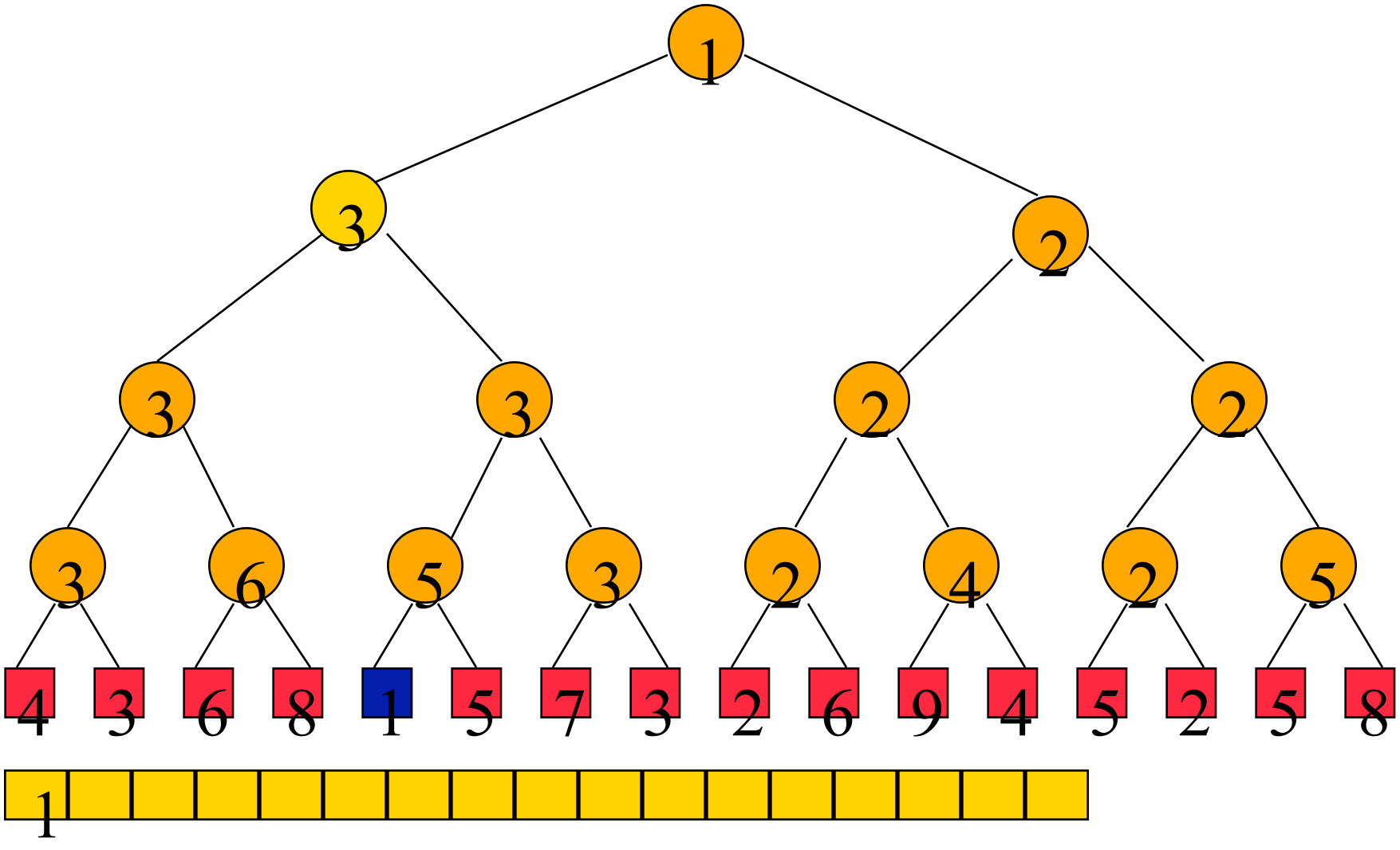

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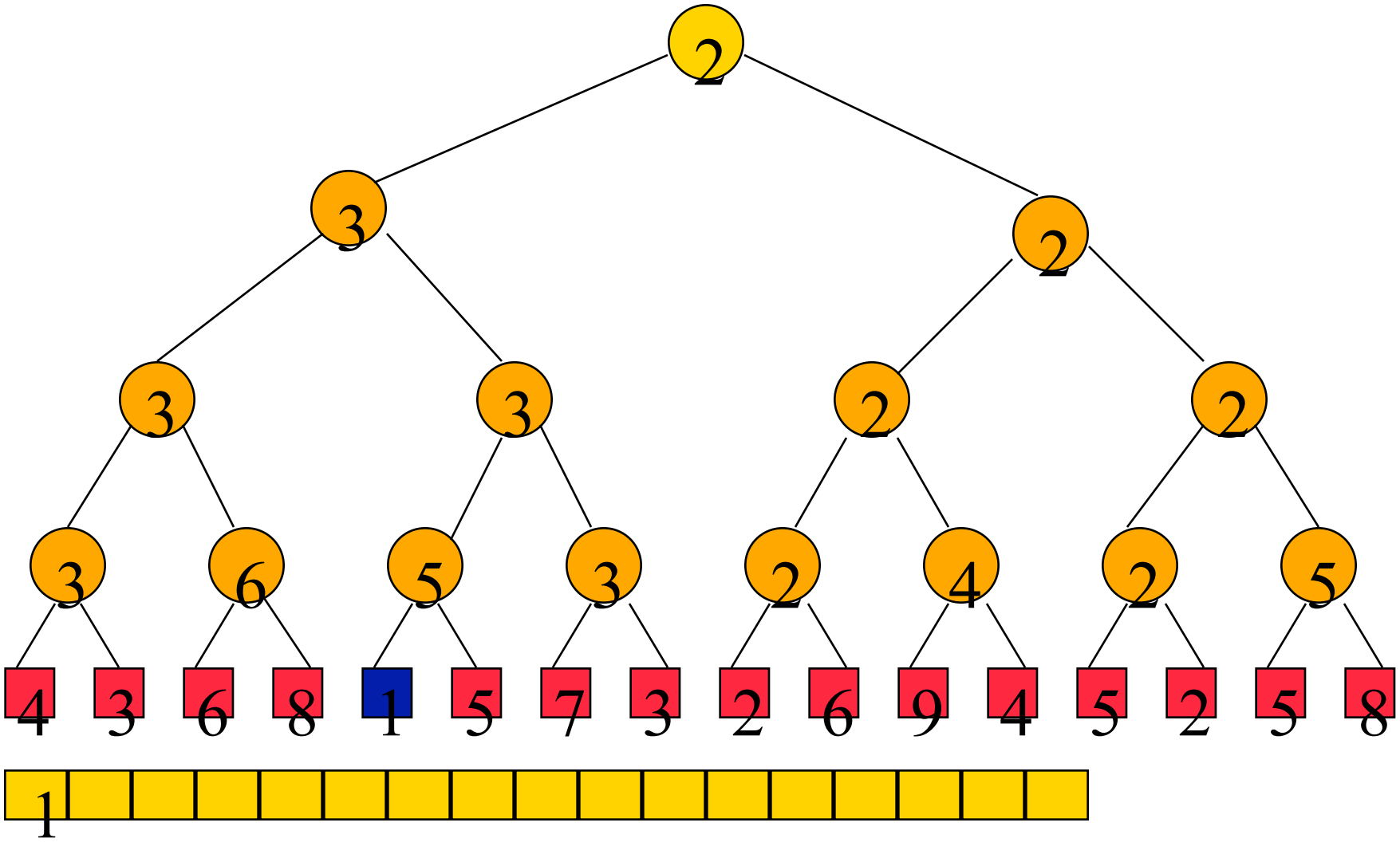

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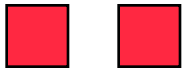

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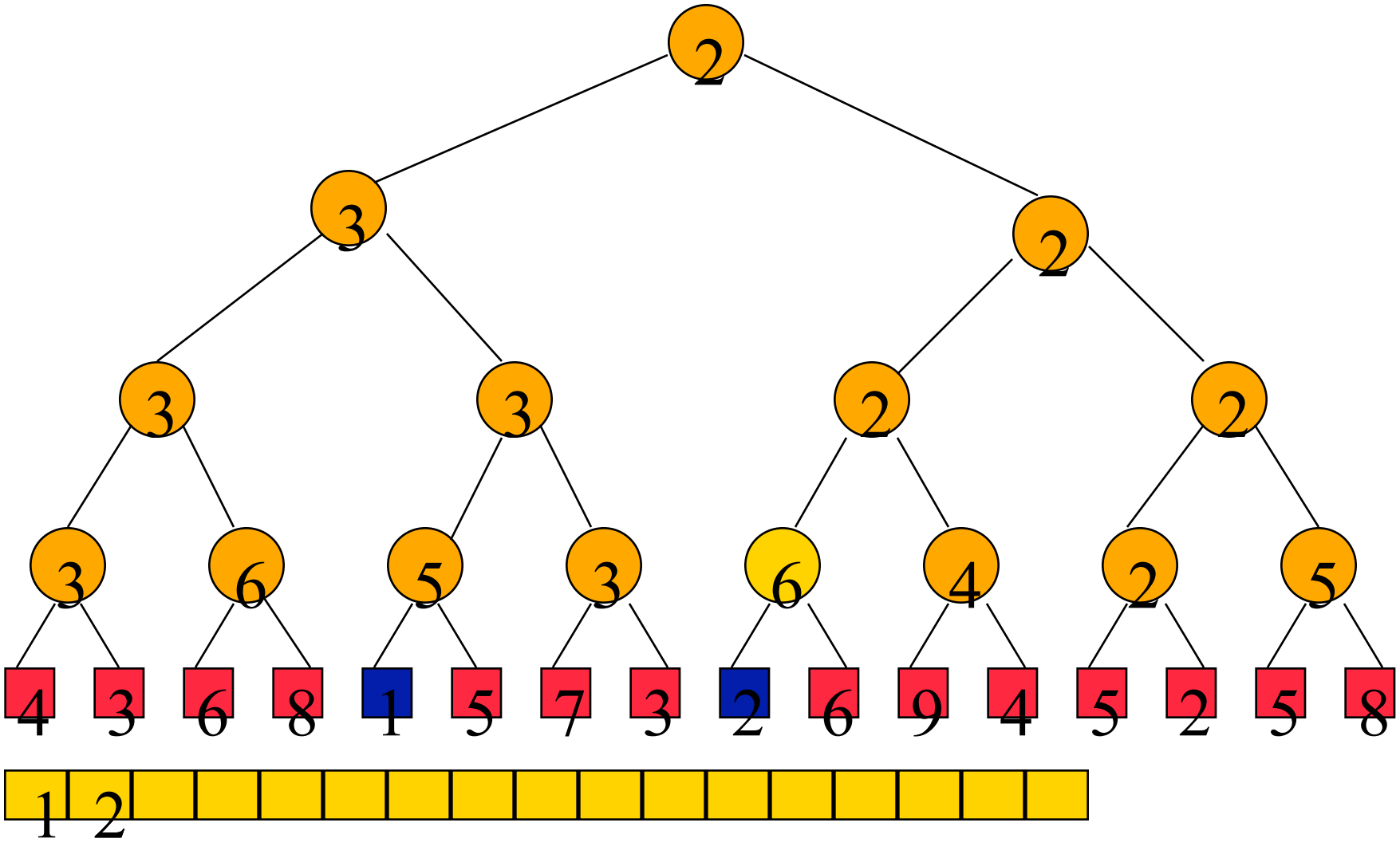

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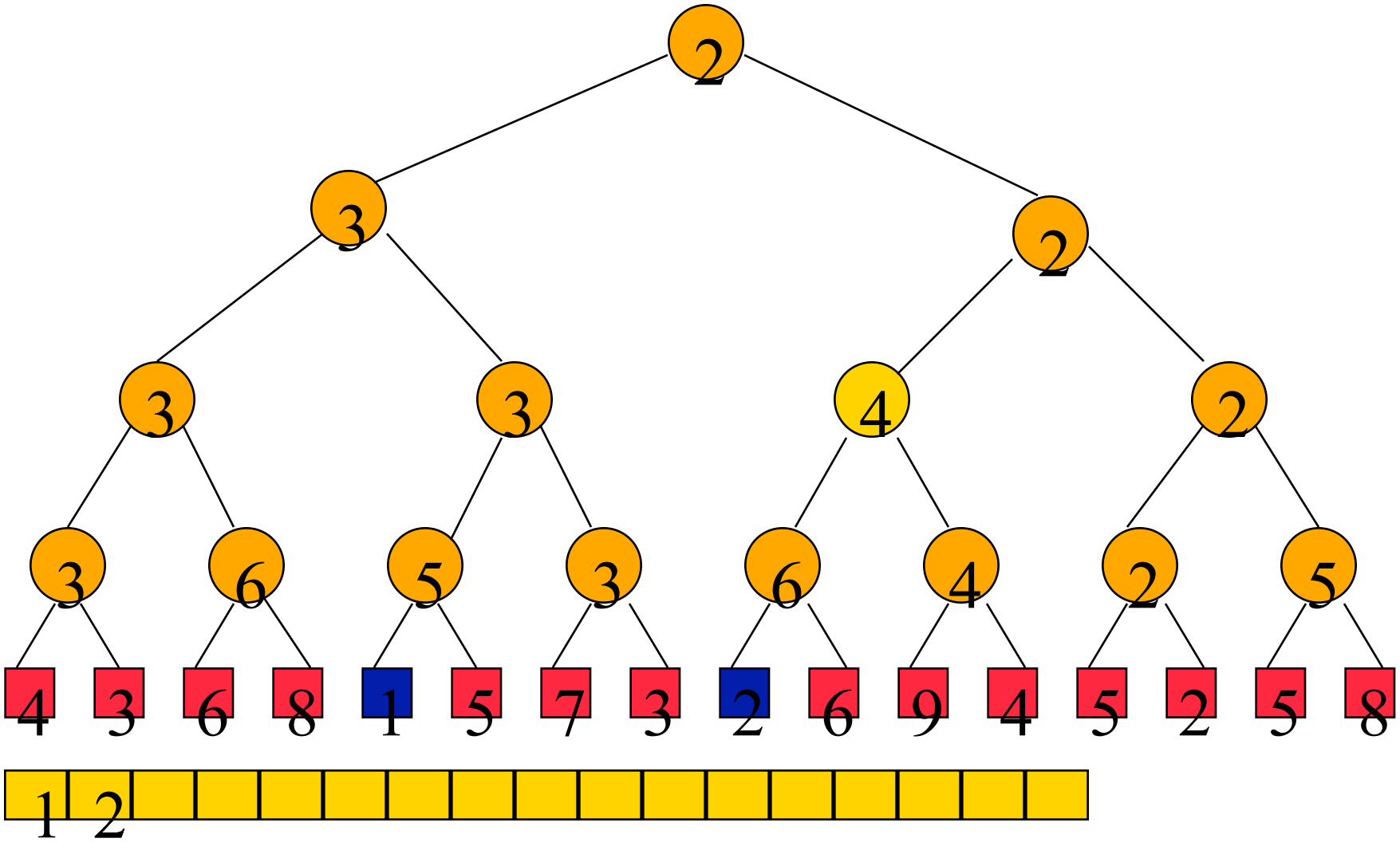

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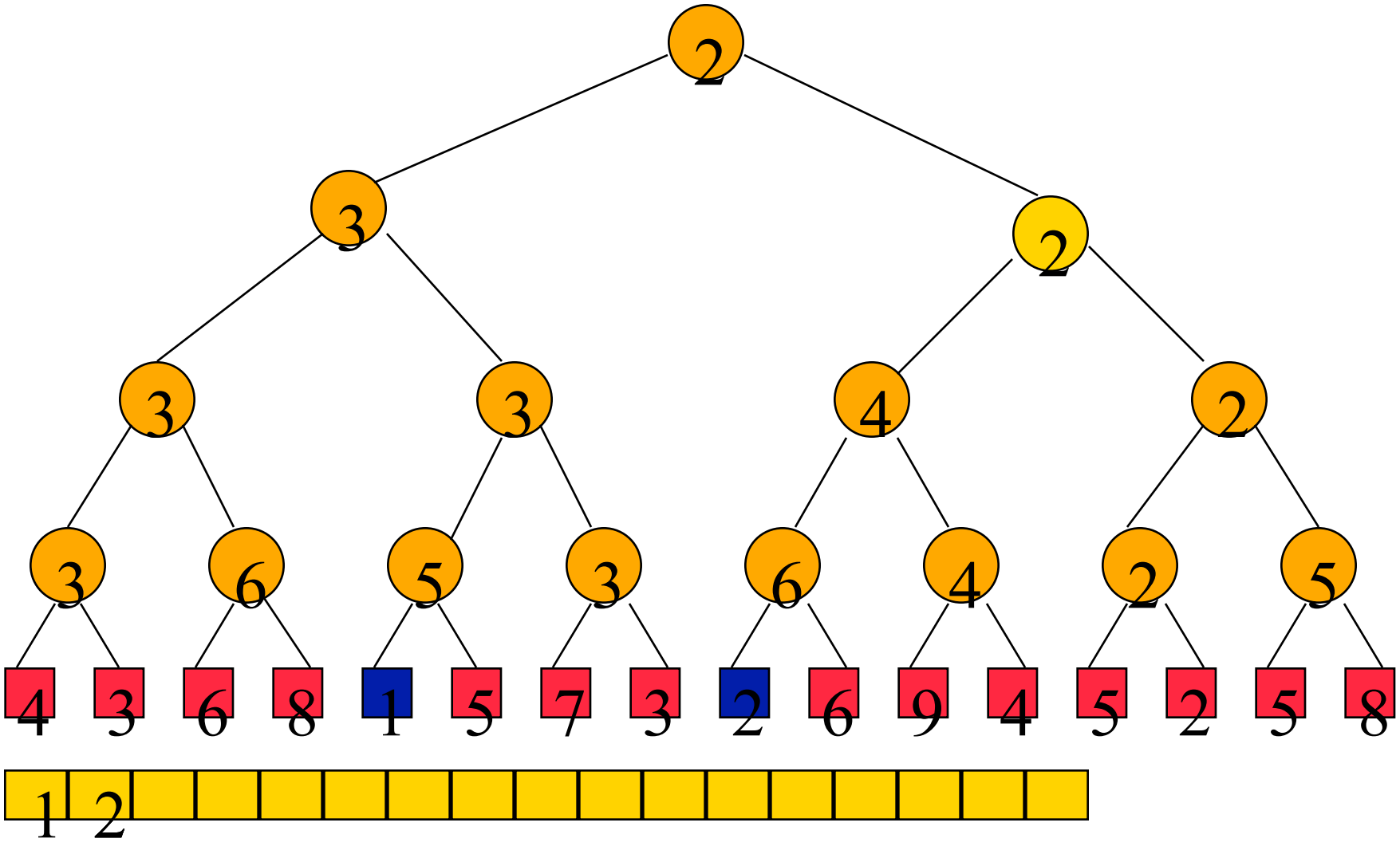

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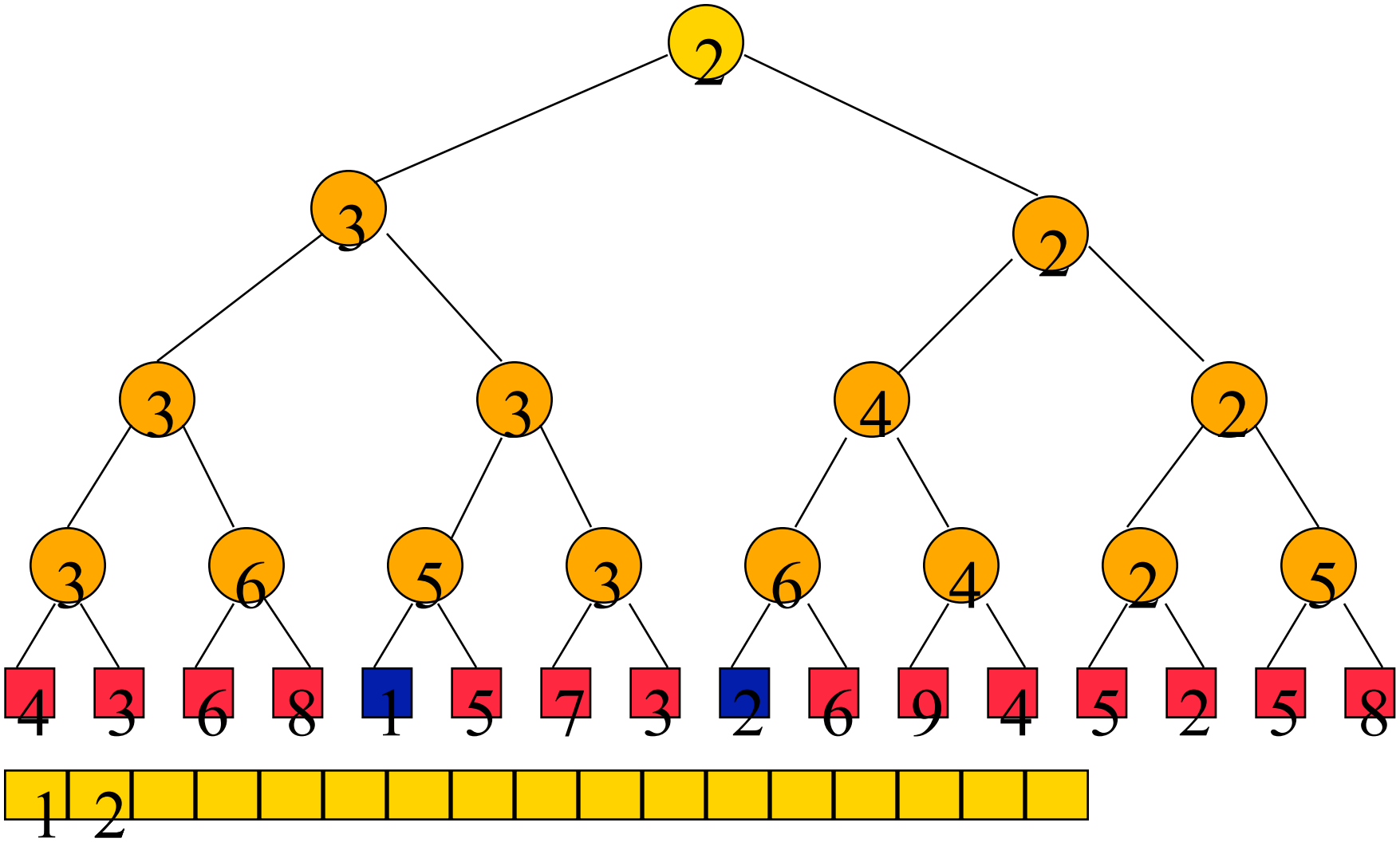

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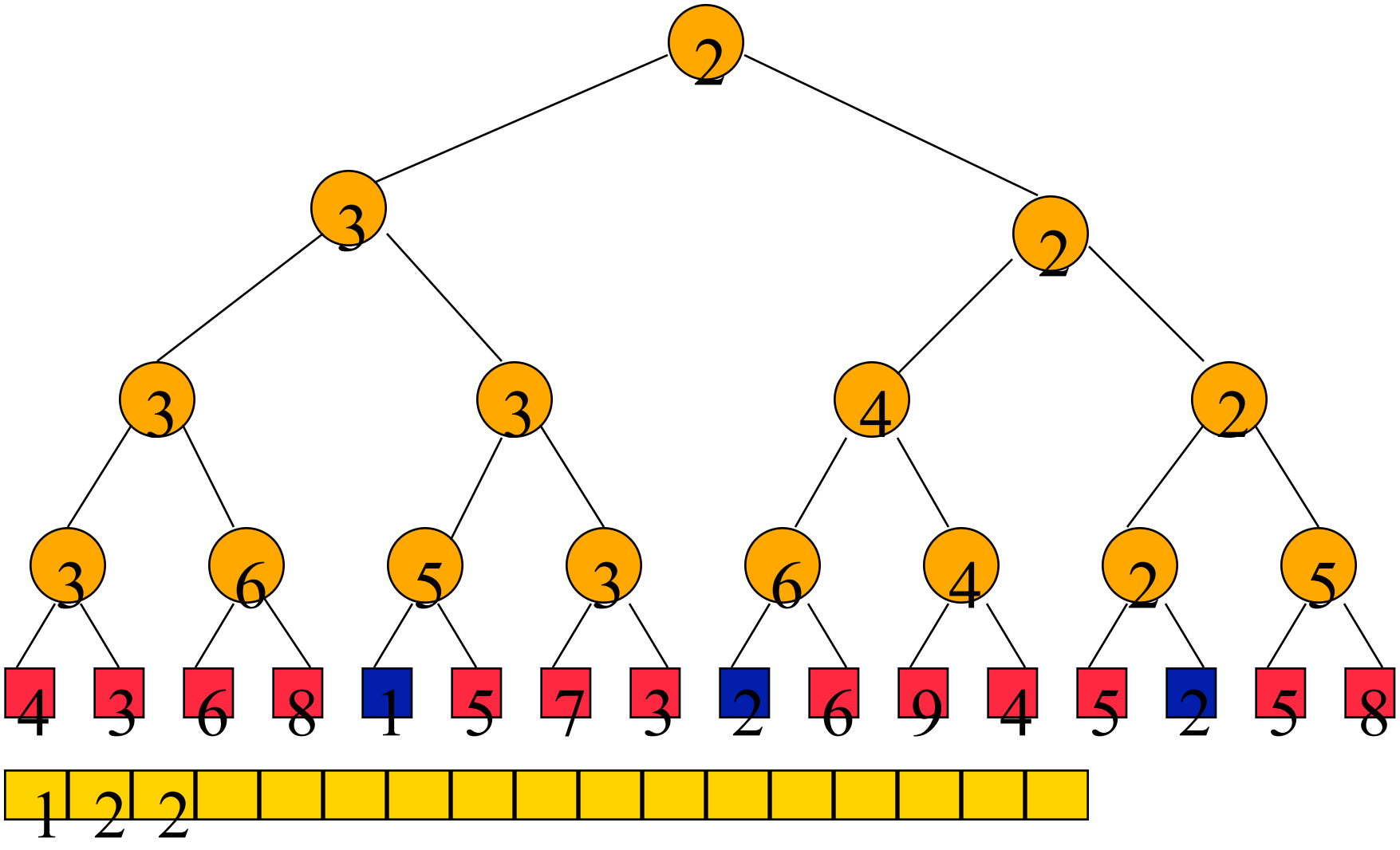

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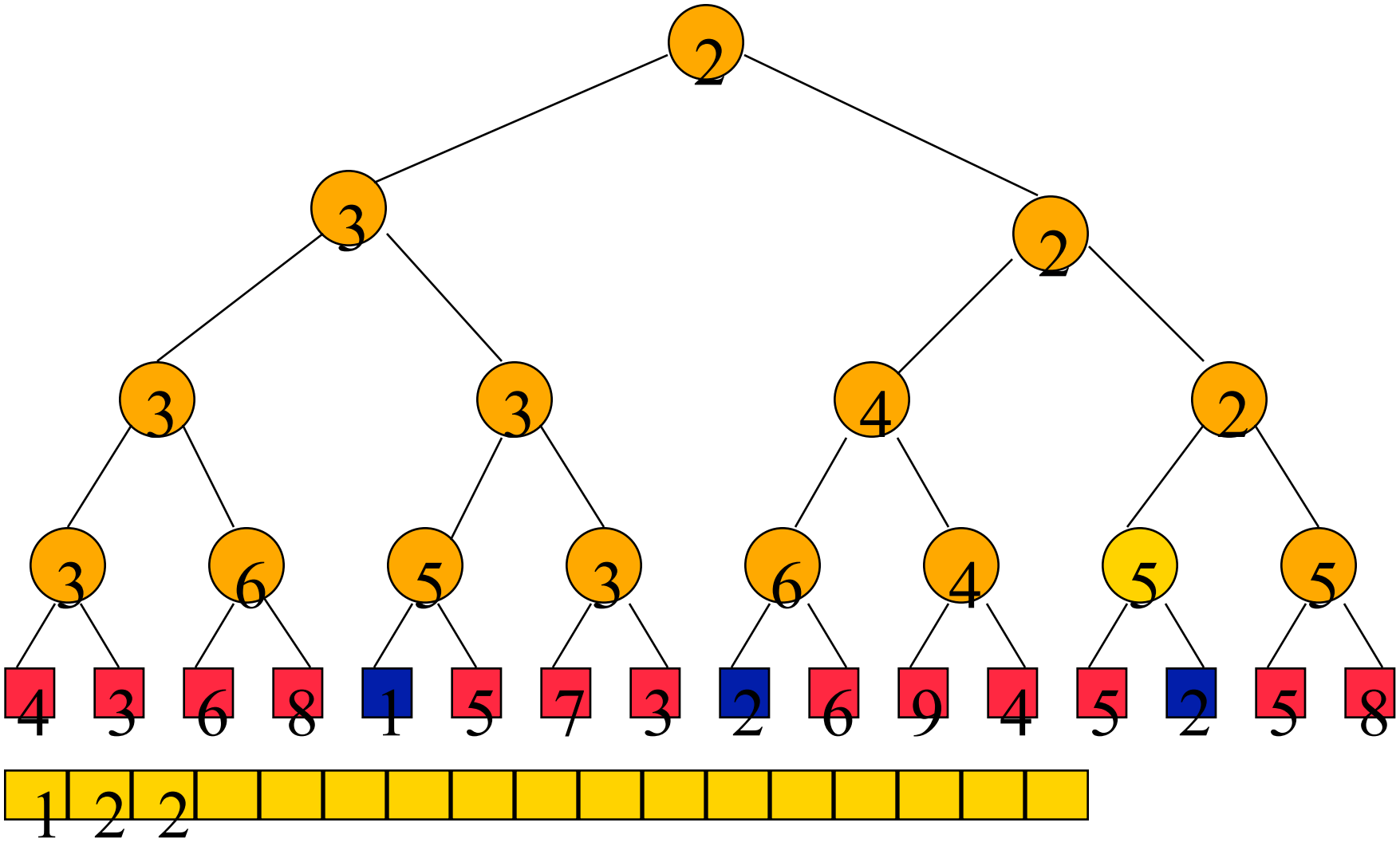

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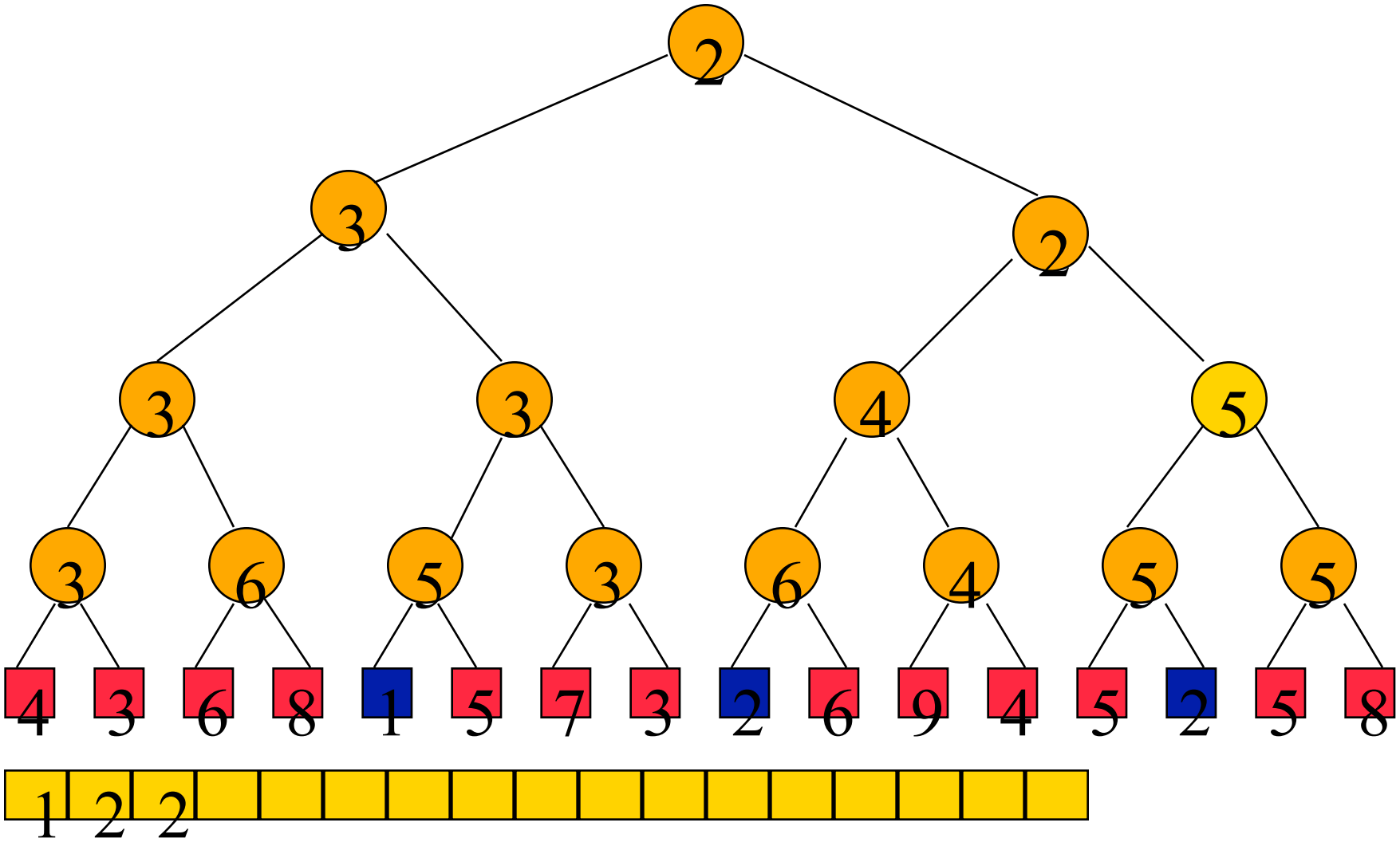

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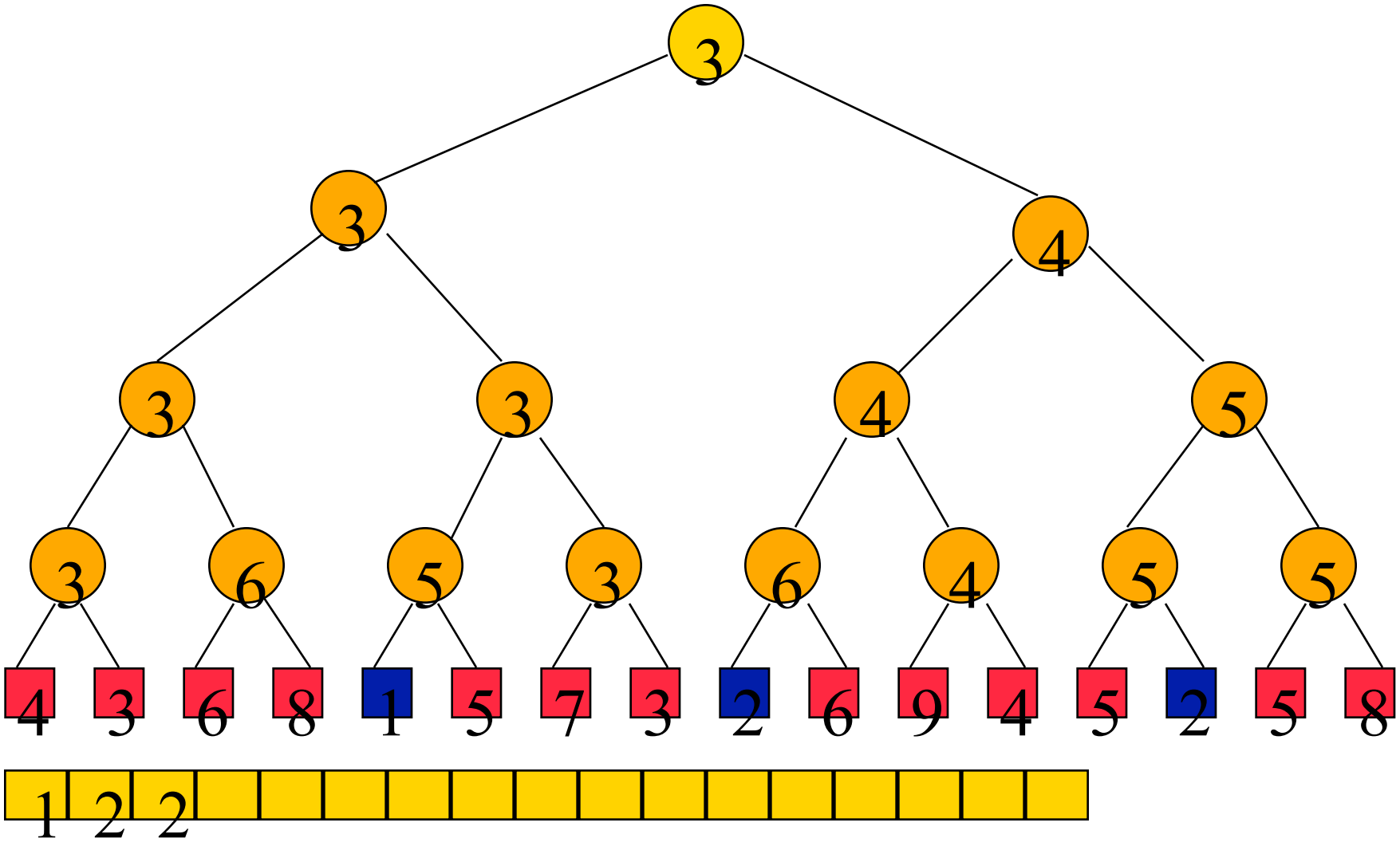

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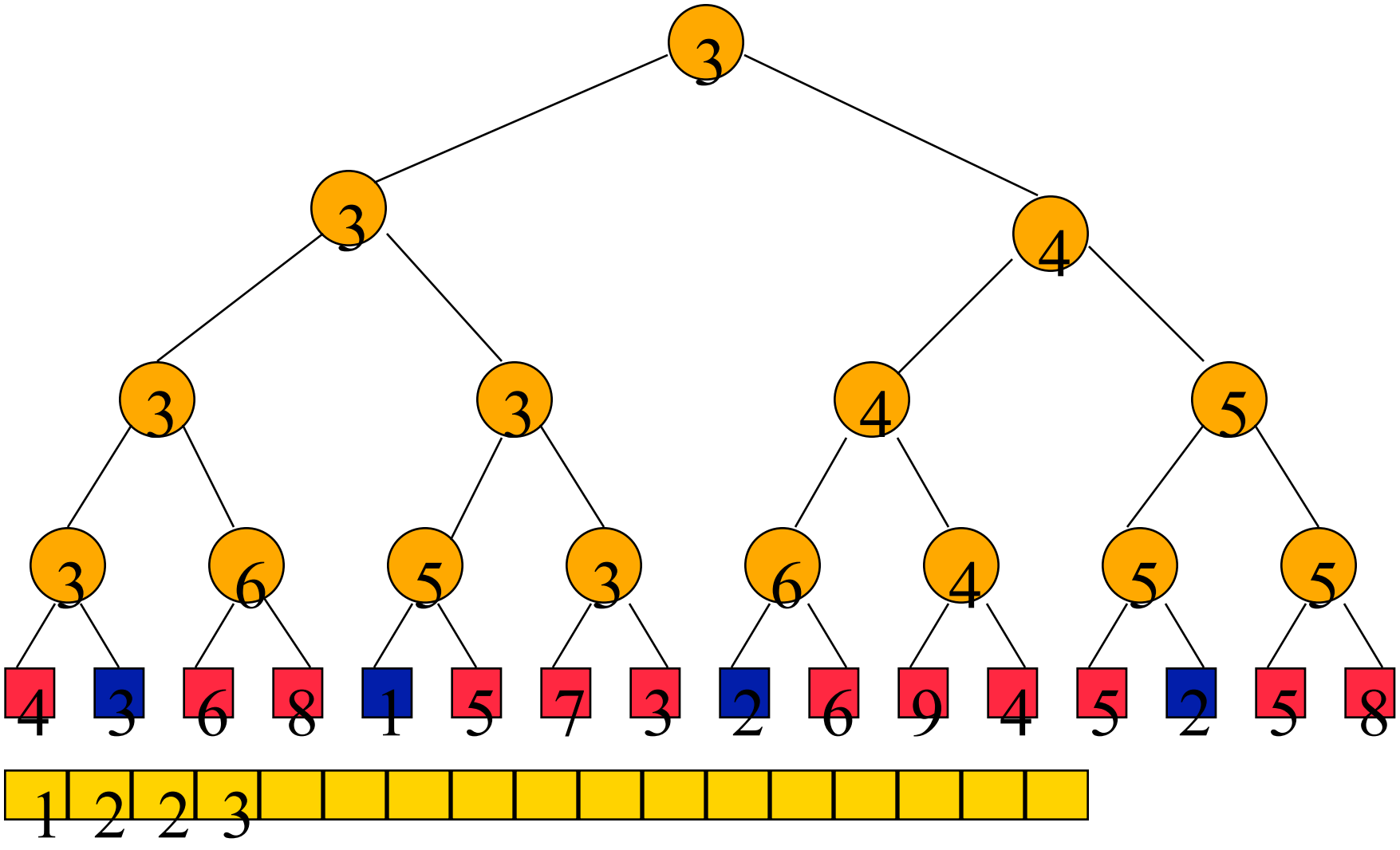

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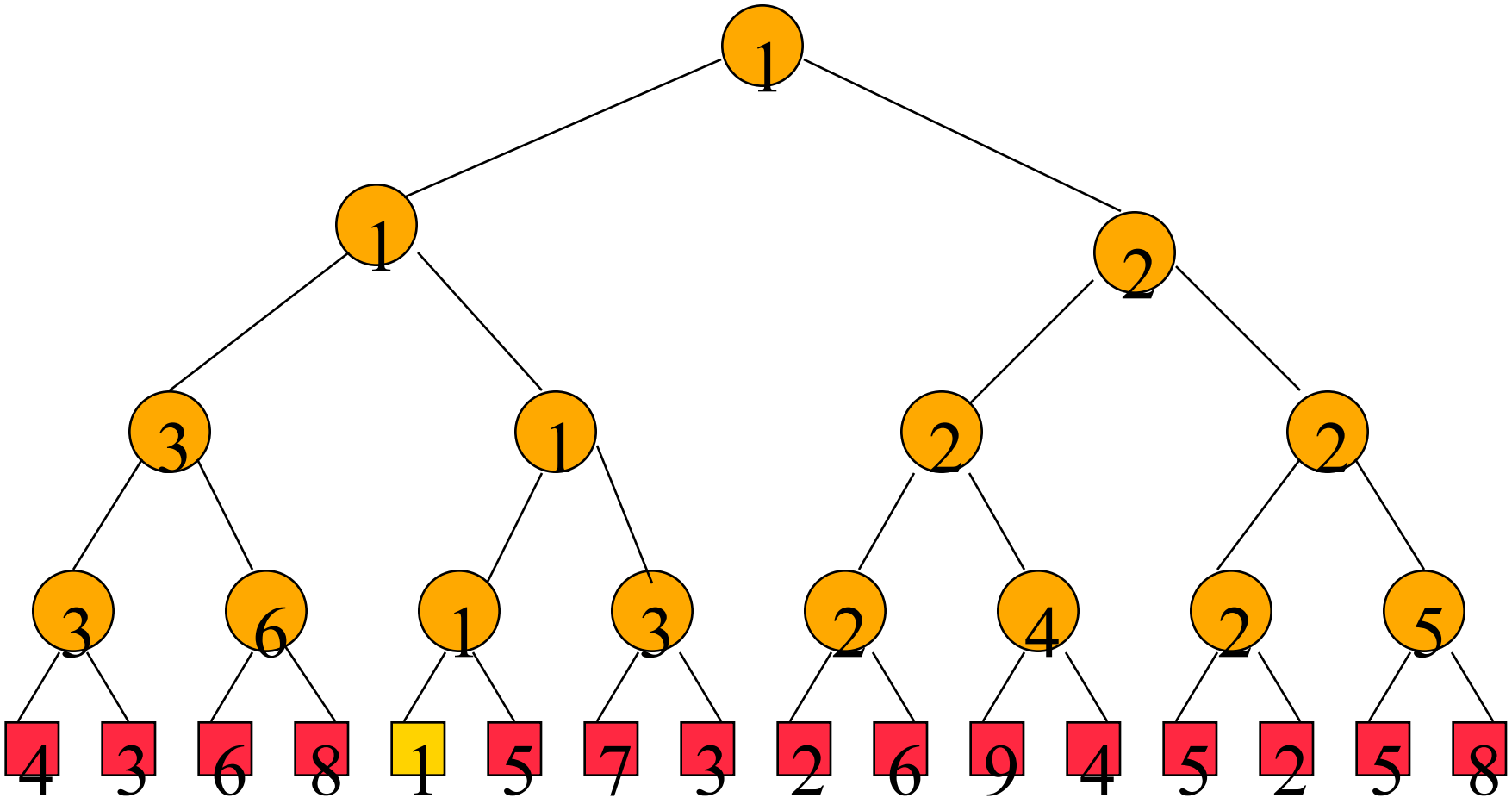

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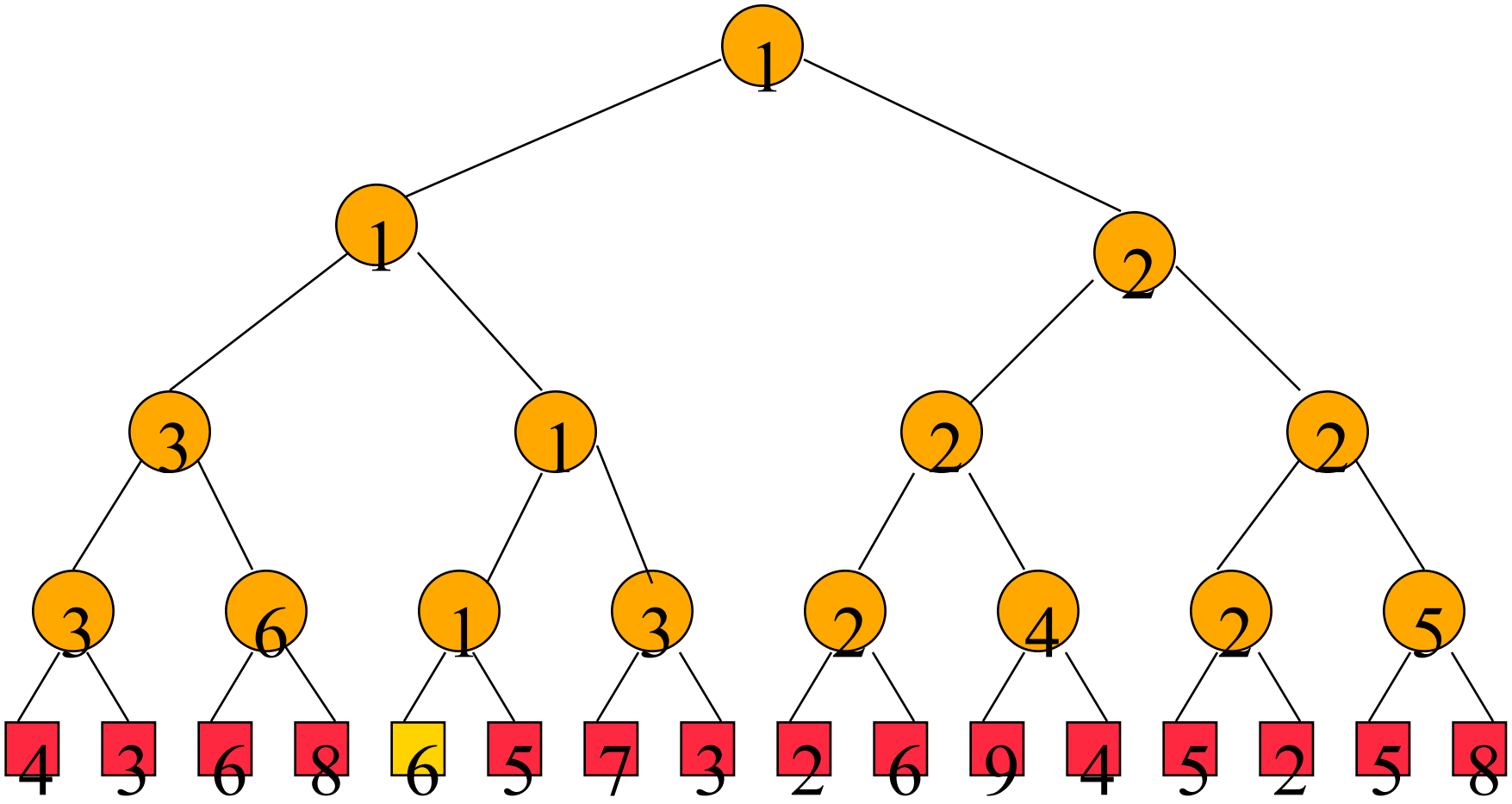
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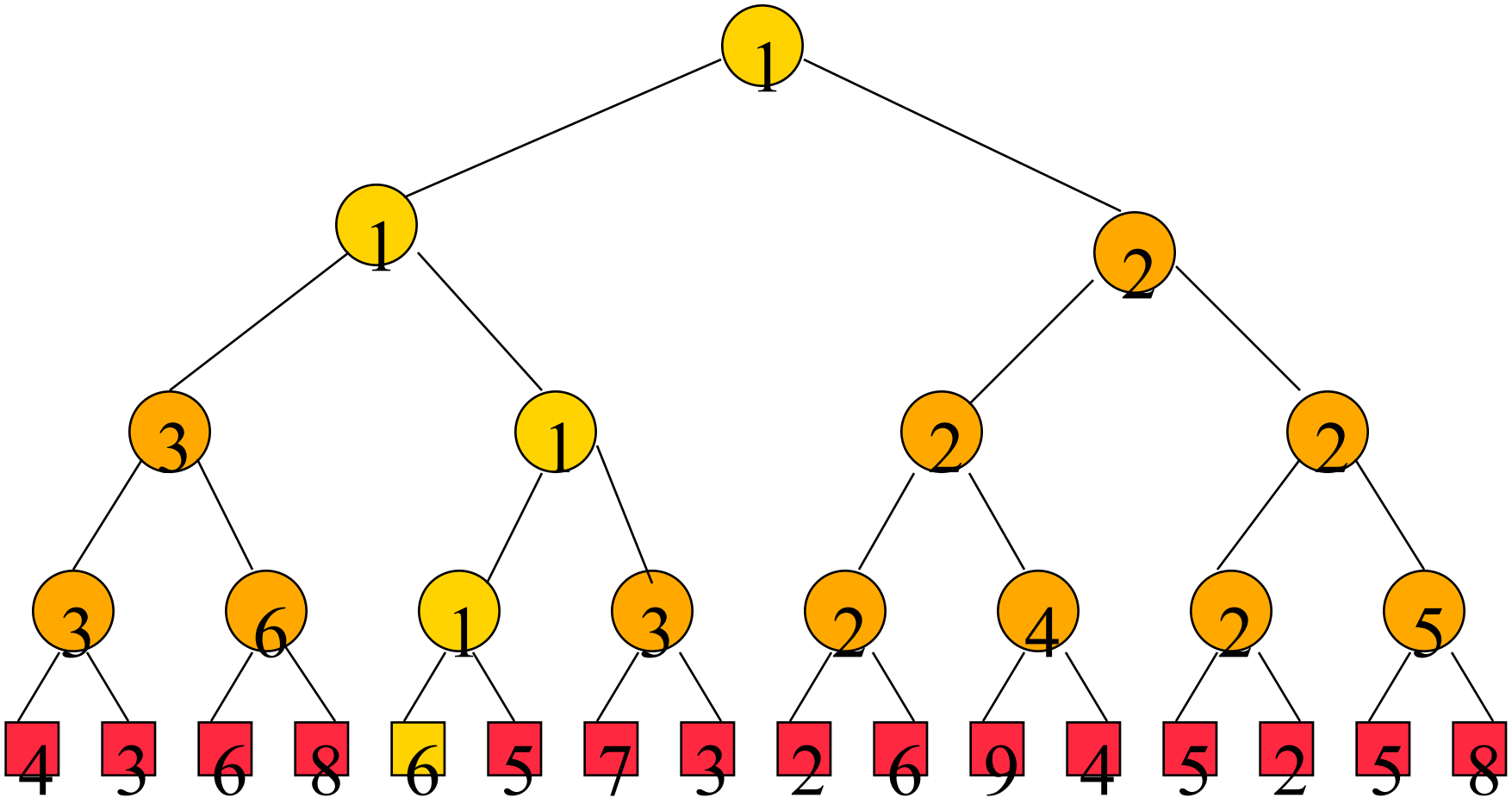
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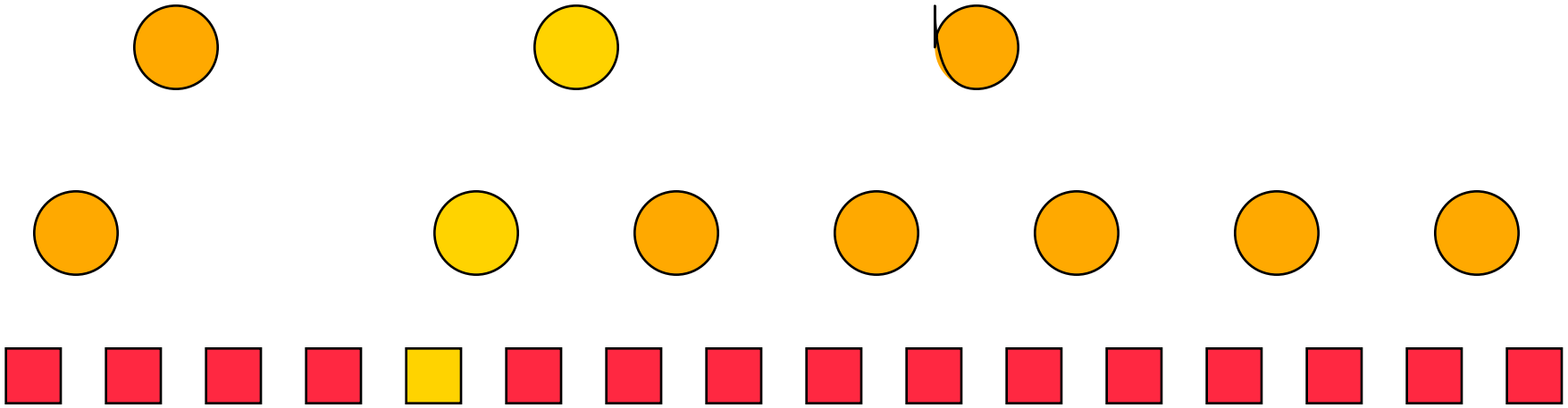

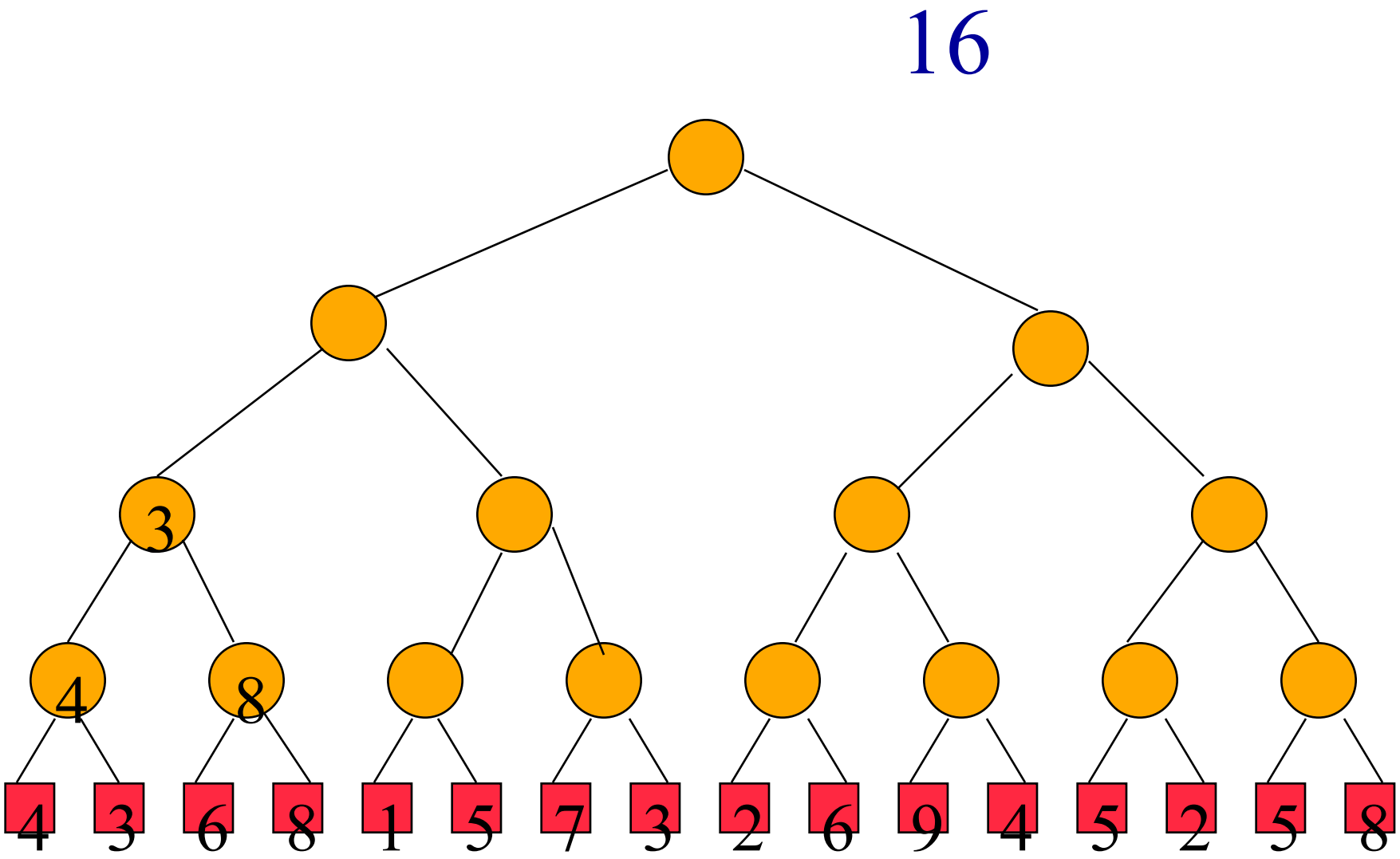
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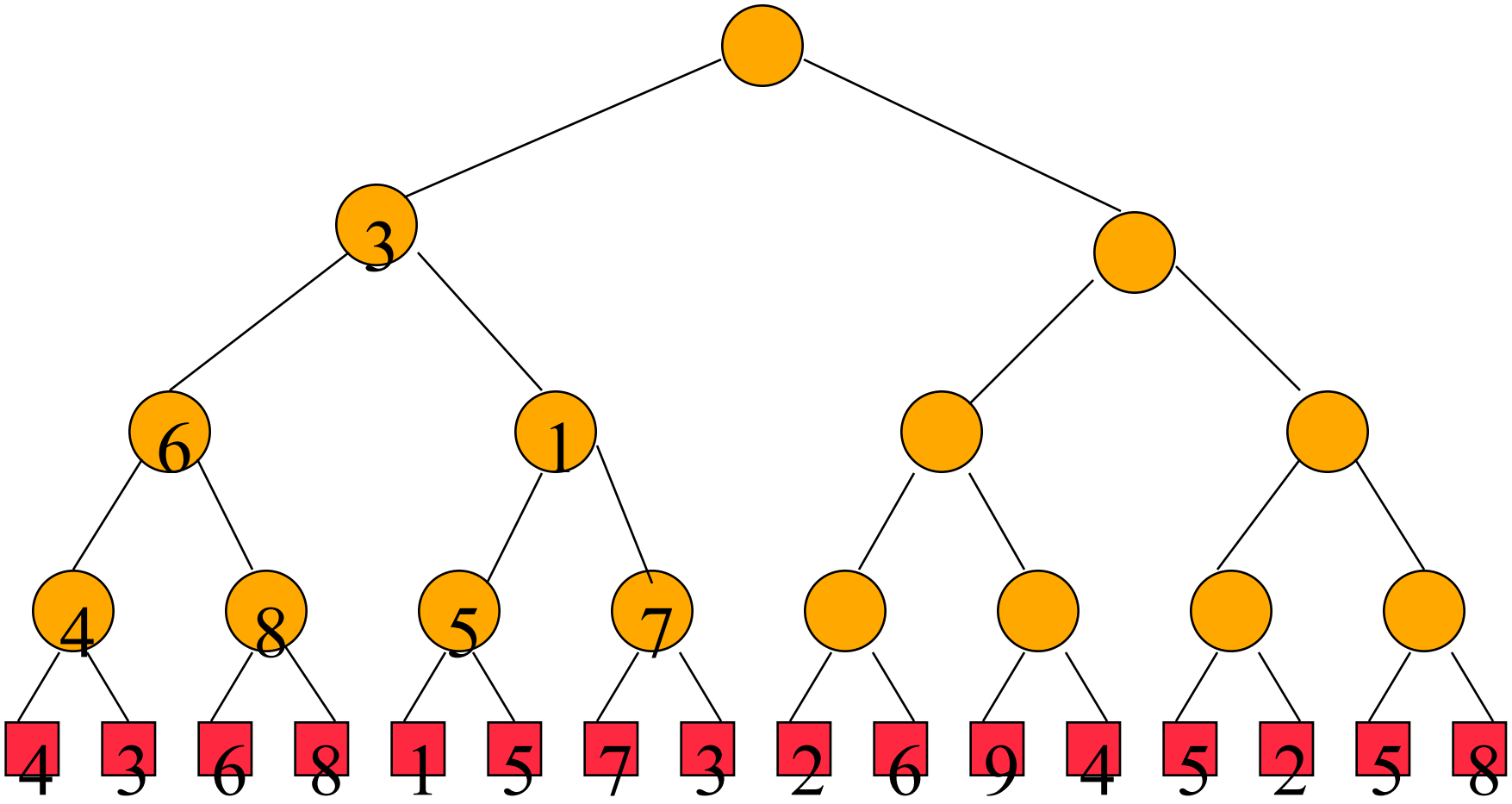
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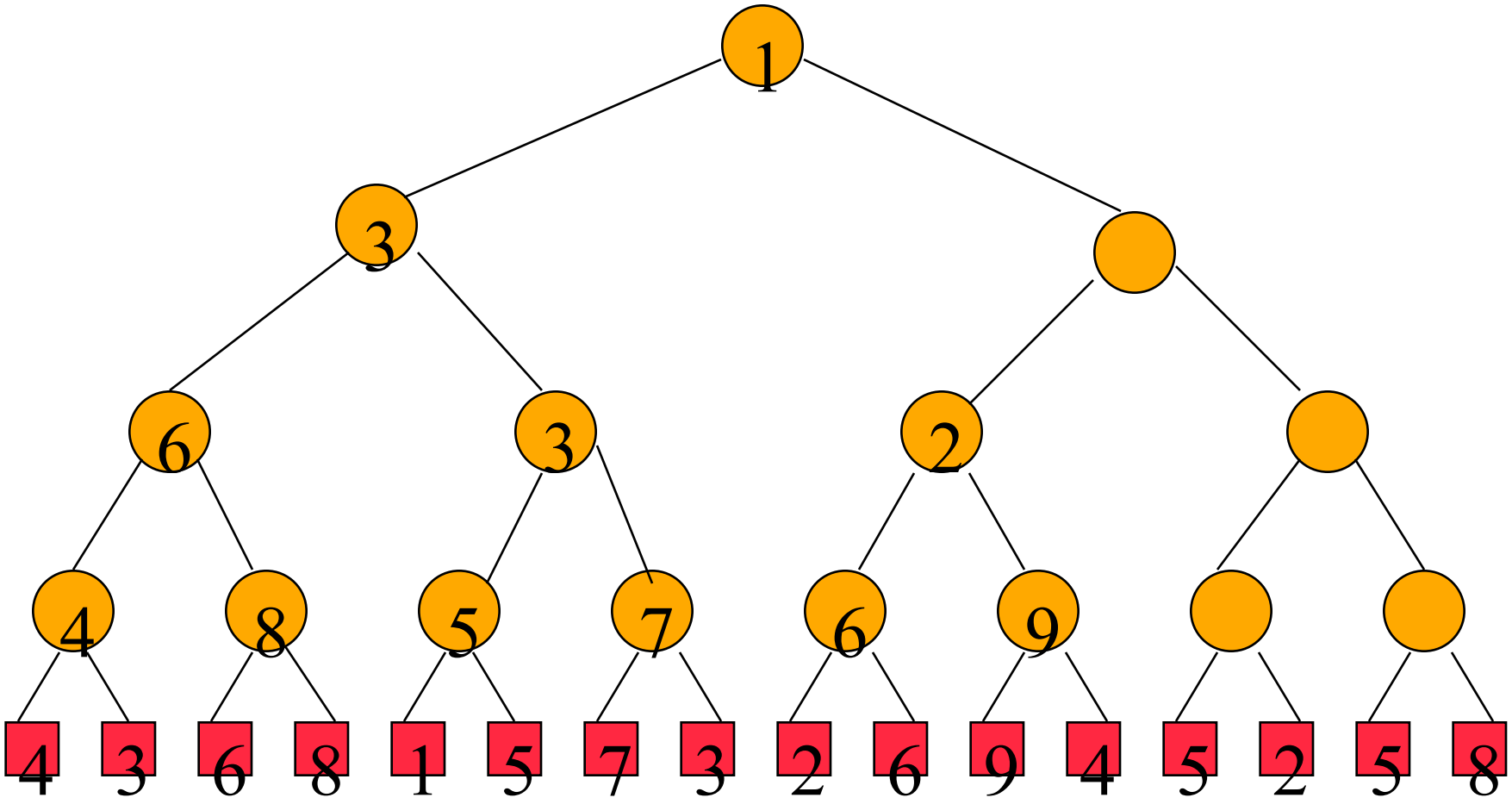
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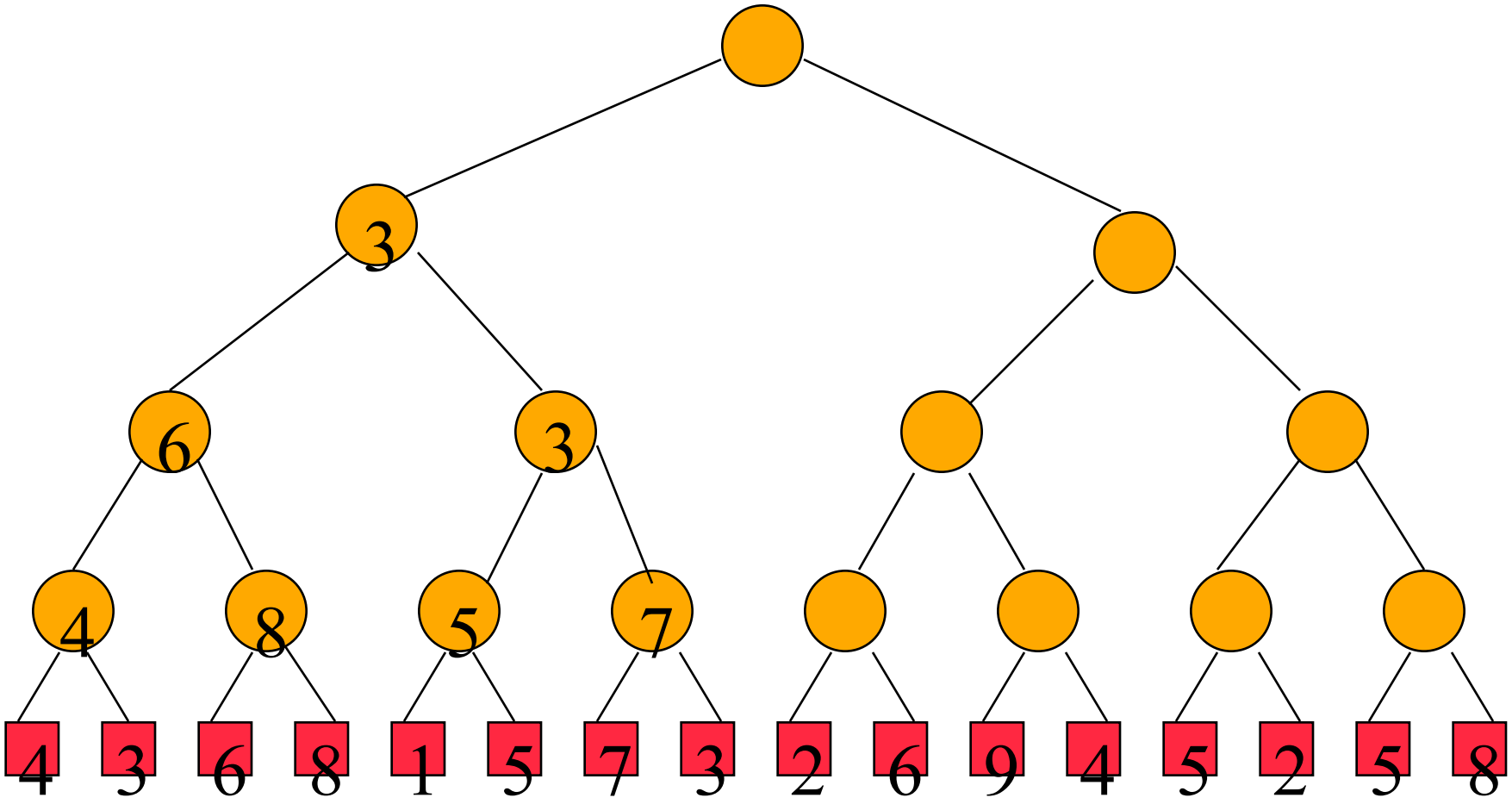
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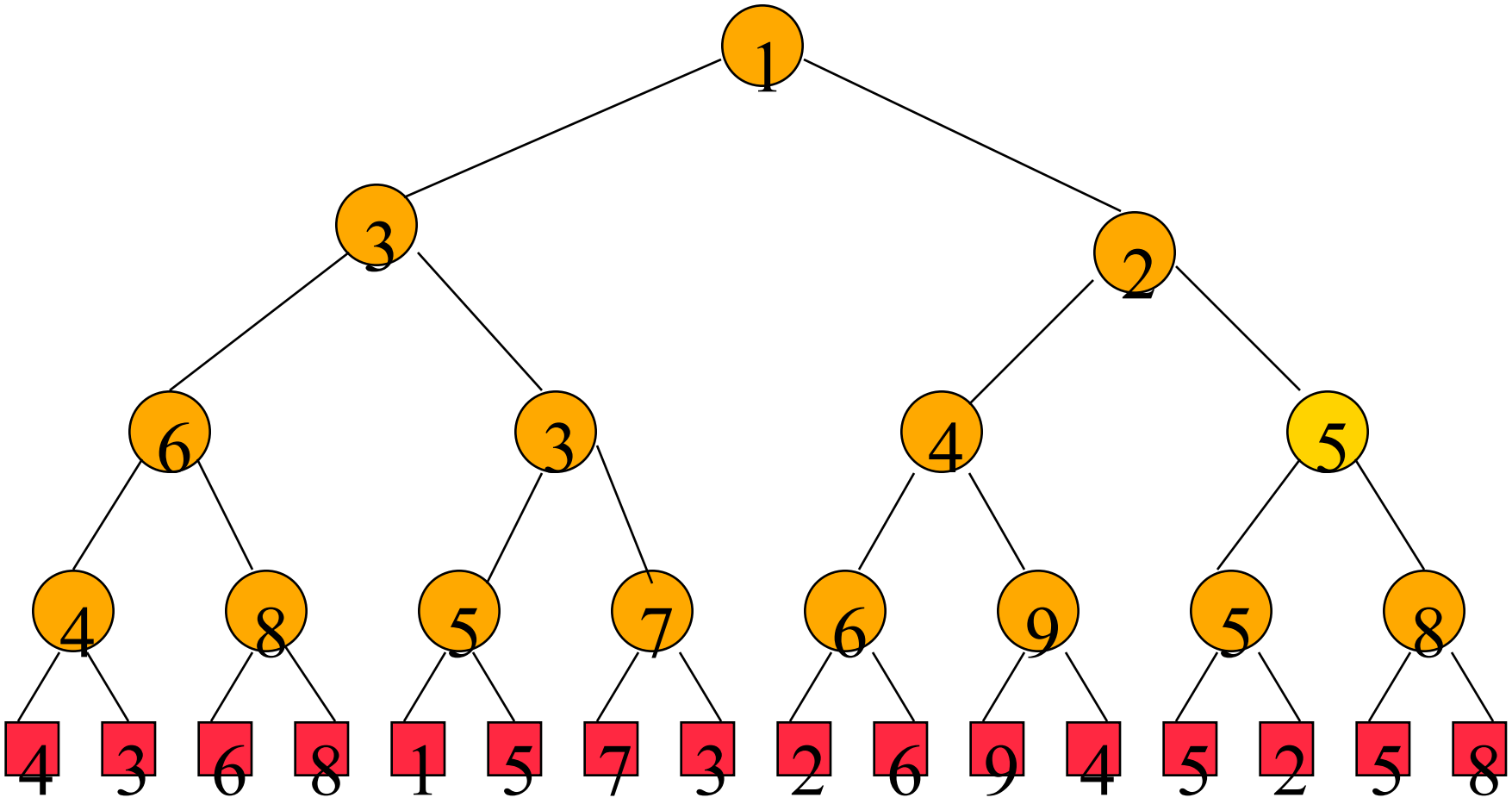
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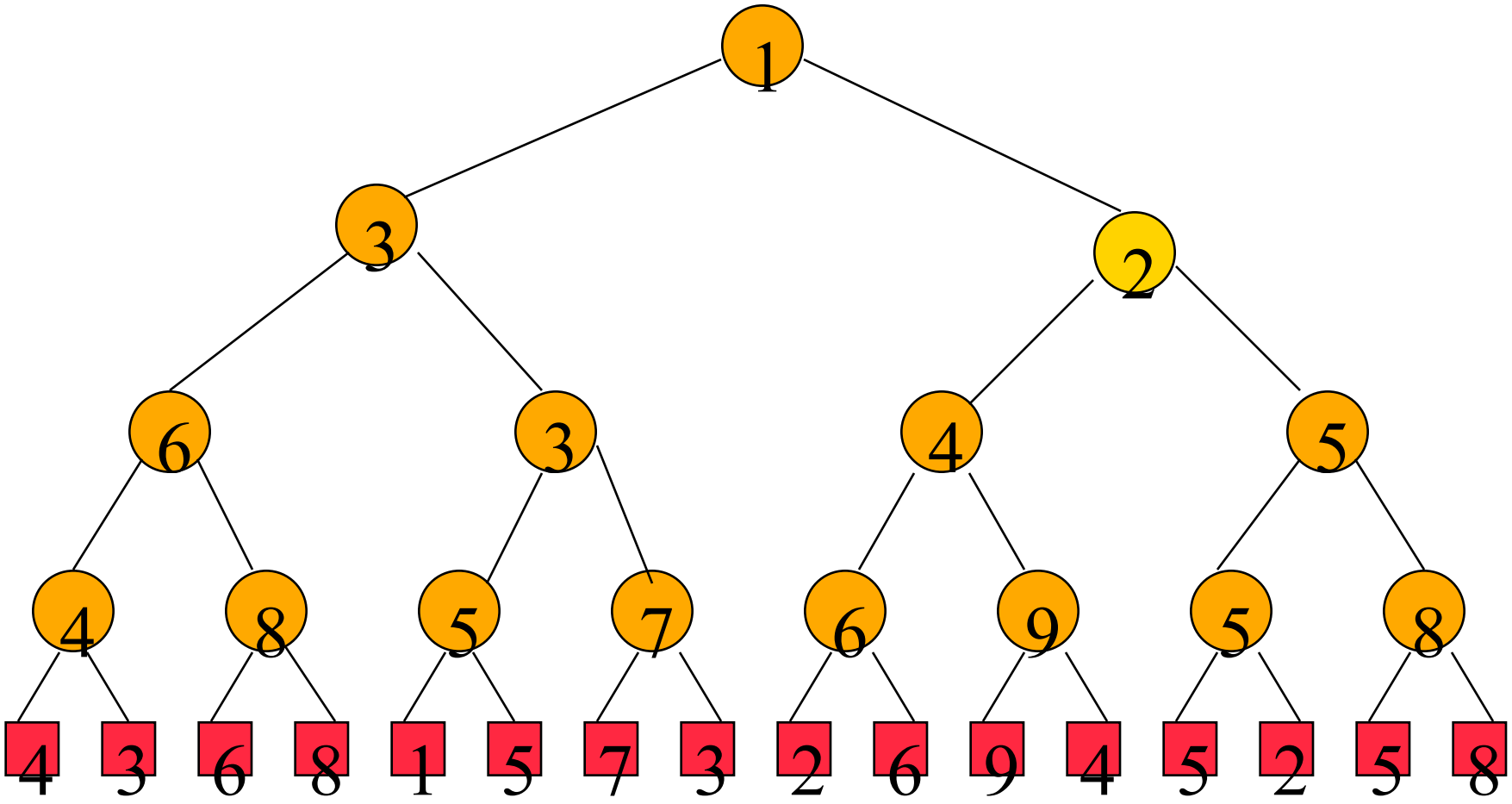
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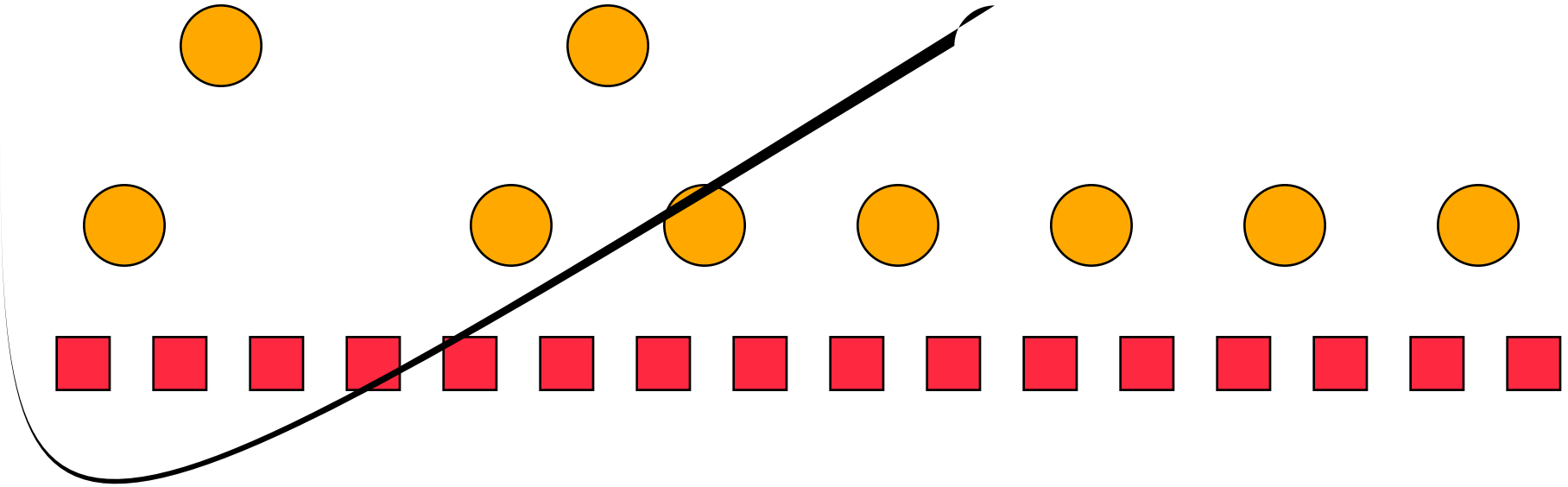

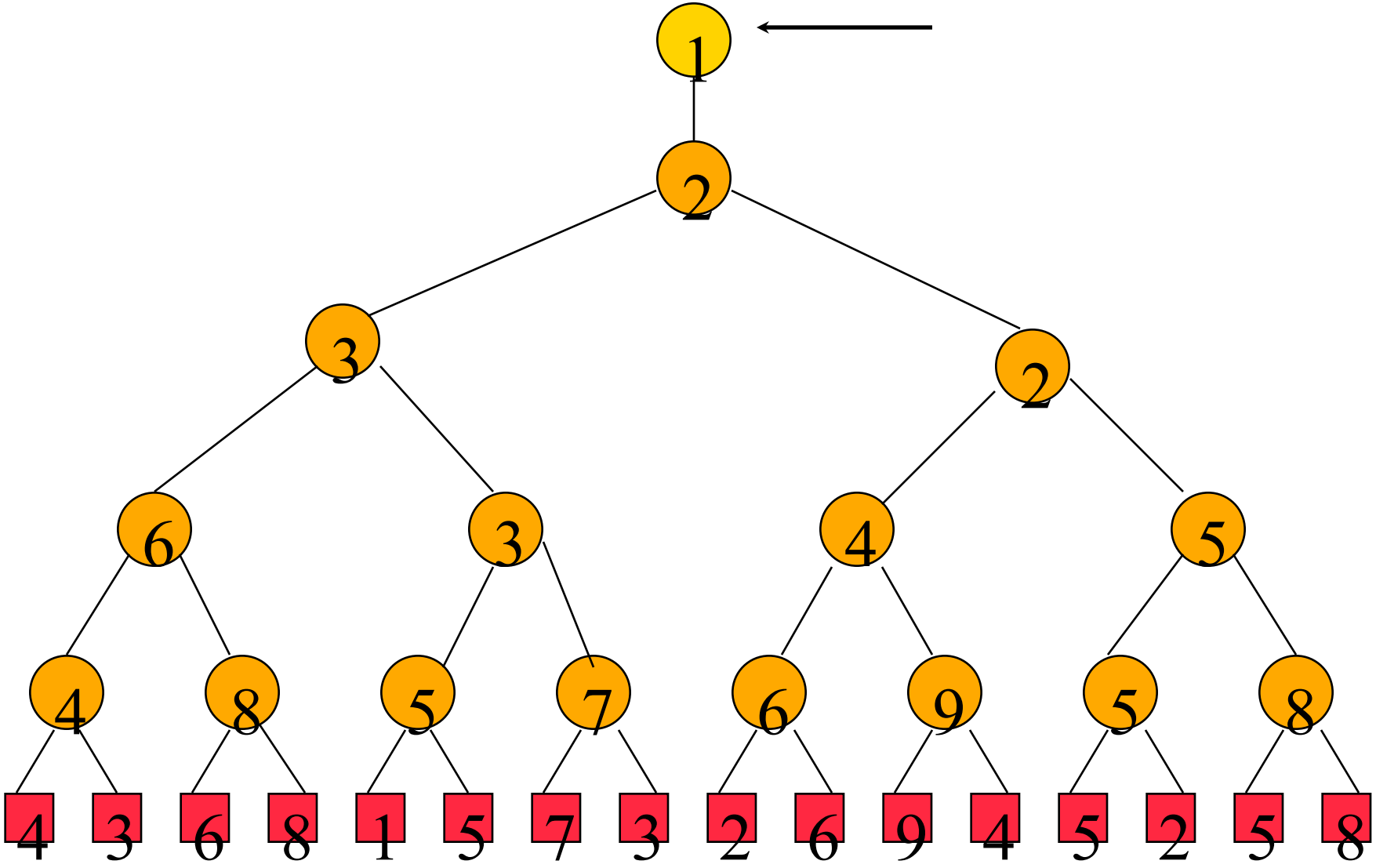
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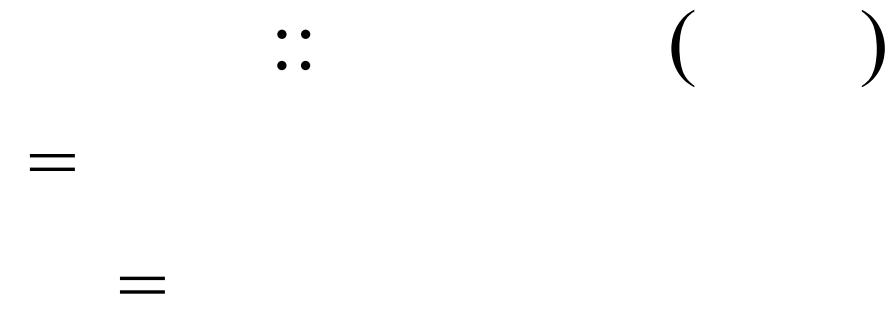

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$2, 5, 6, 3, 4$

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$1 = 2, 5, 3$

$2 = 6, 4$

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$= 4, 7, 3, 6$

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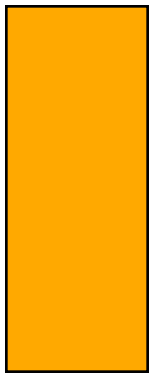

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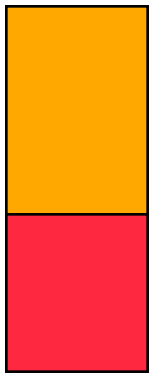

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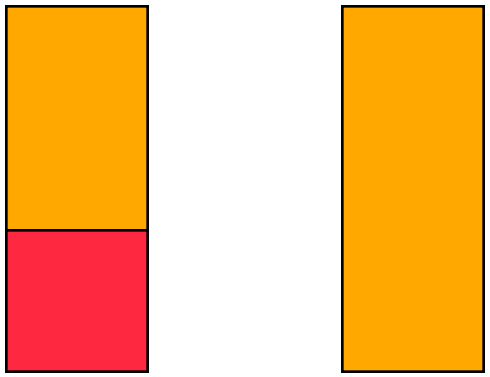

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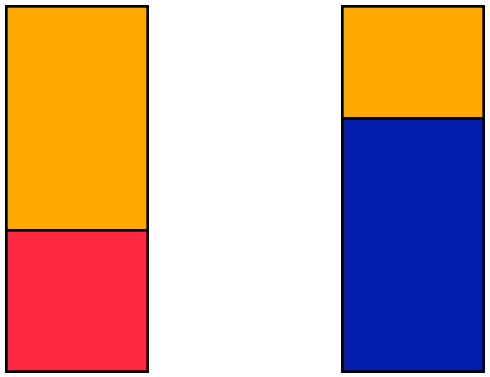

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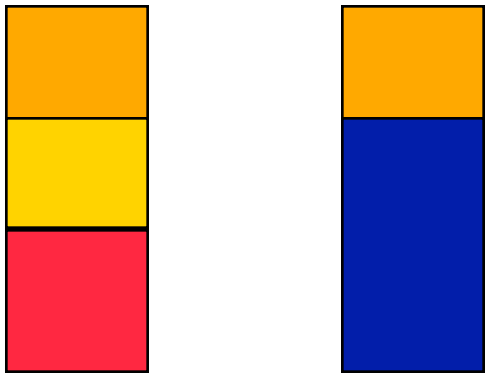

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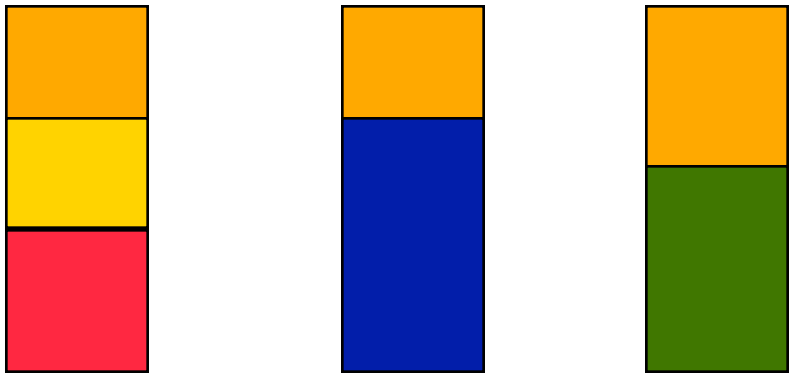

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$= [4, 7, 3, 6]$

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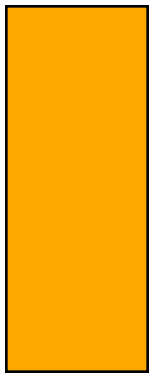

Pack red item into first bin.

$= 4$

$= [4, 7, 3, 6]$

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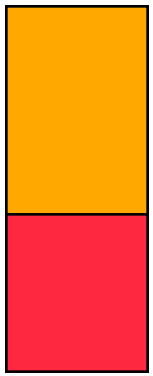

Pack blue item next.

Doesn't fit, so start a new bin.

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$= [4, 7, 3, 6]$

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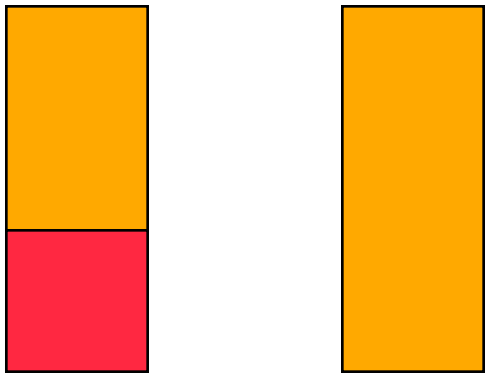

$= 4$

$= [4, 7, 3, 6]$

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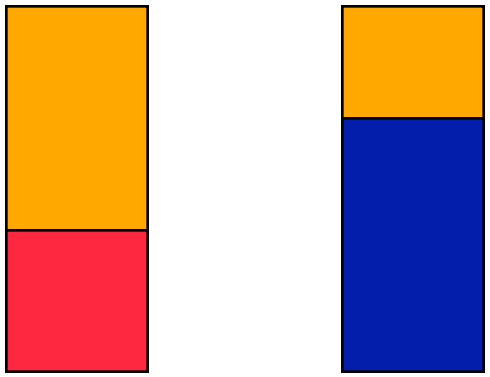

Pack yellow item into second bin.

$= 4$

$= [4, 7, 3, 6]$

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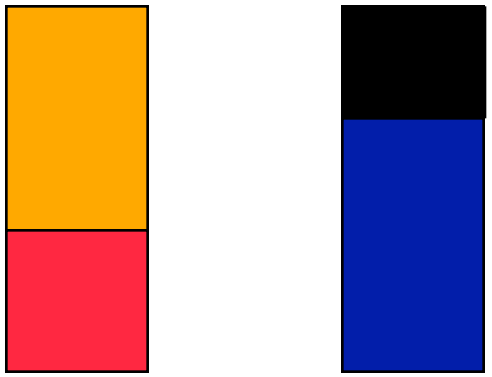

Pack green item into first bin.

$= 4$

$= [4, 7, 3, 6]$

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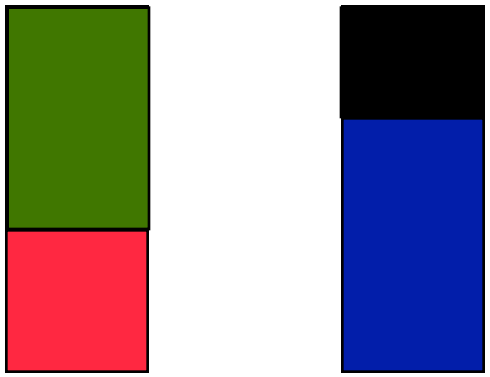

Optimal packing.

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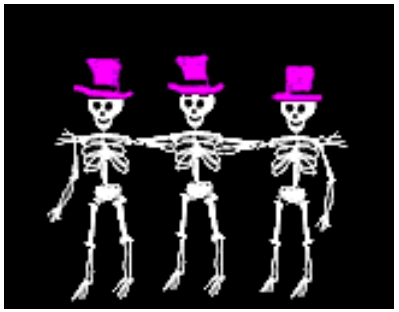

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- **Exercises: P301-1,4**

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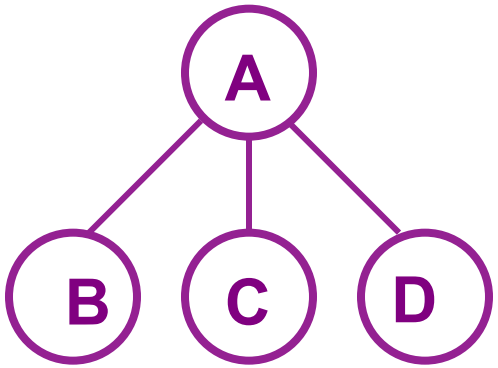

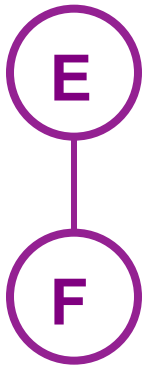

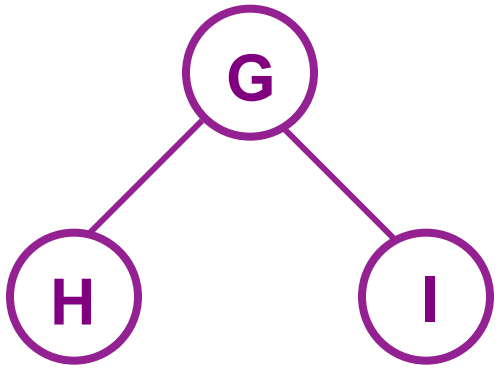

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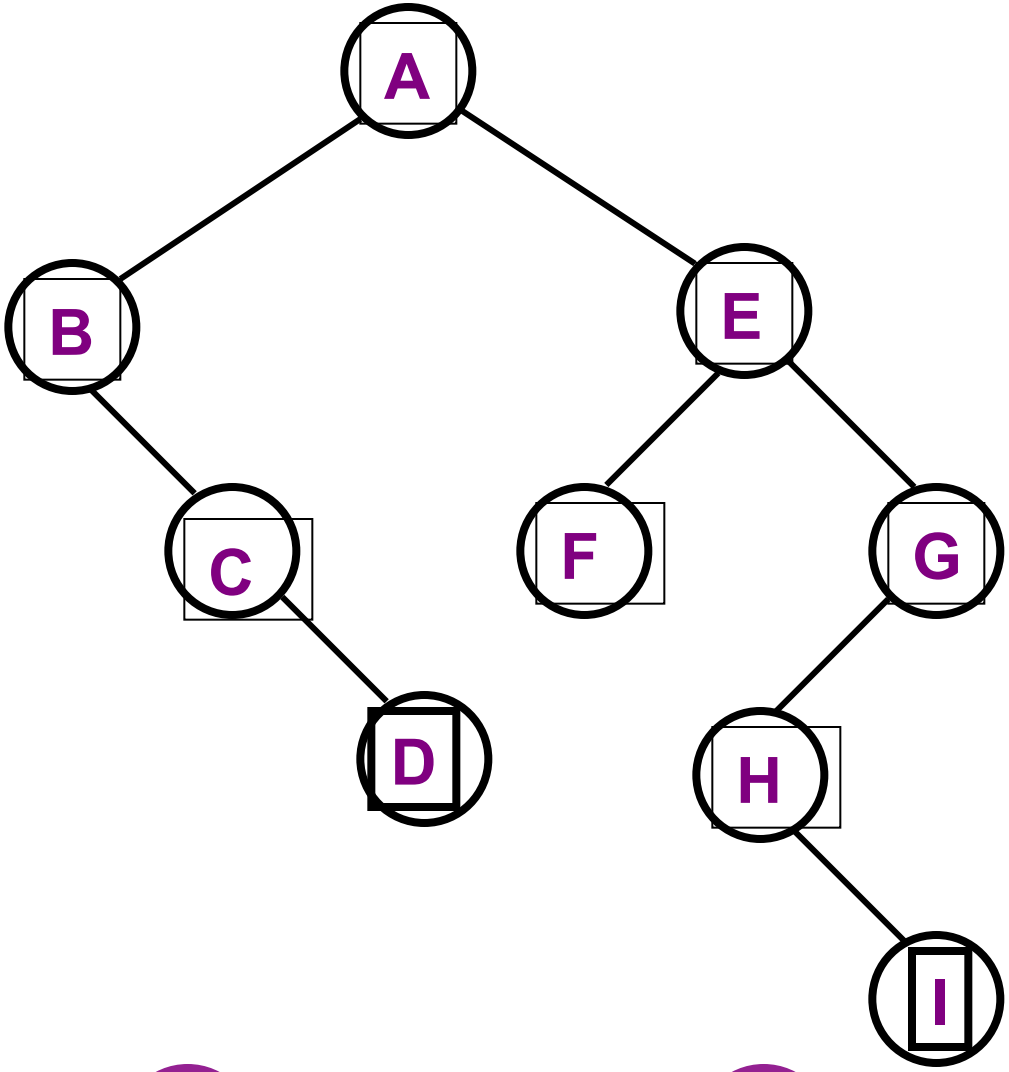
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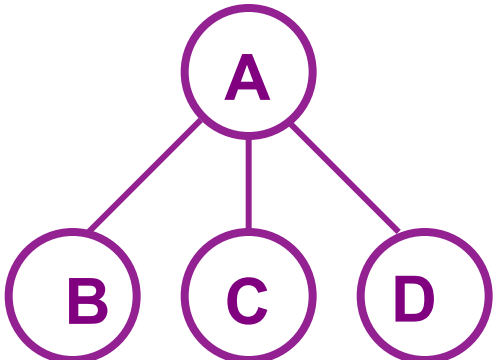
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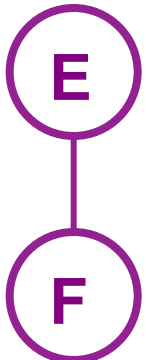
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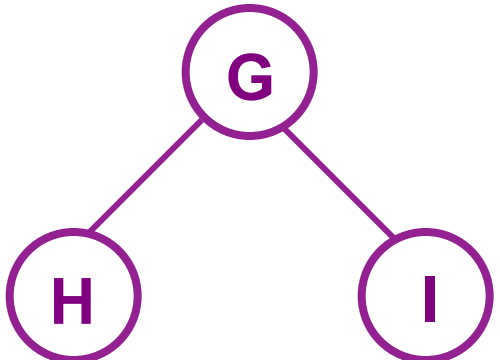
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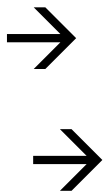

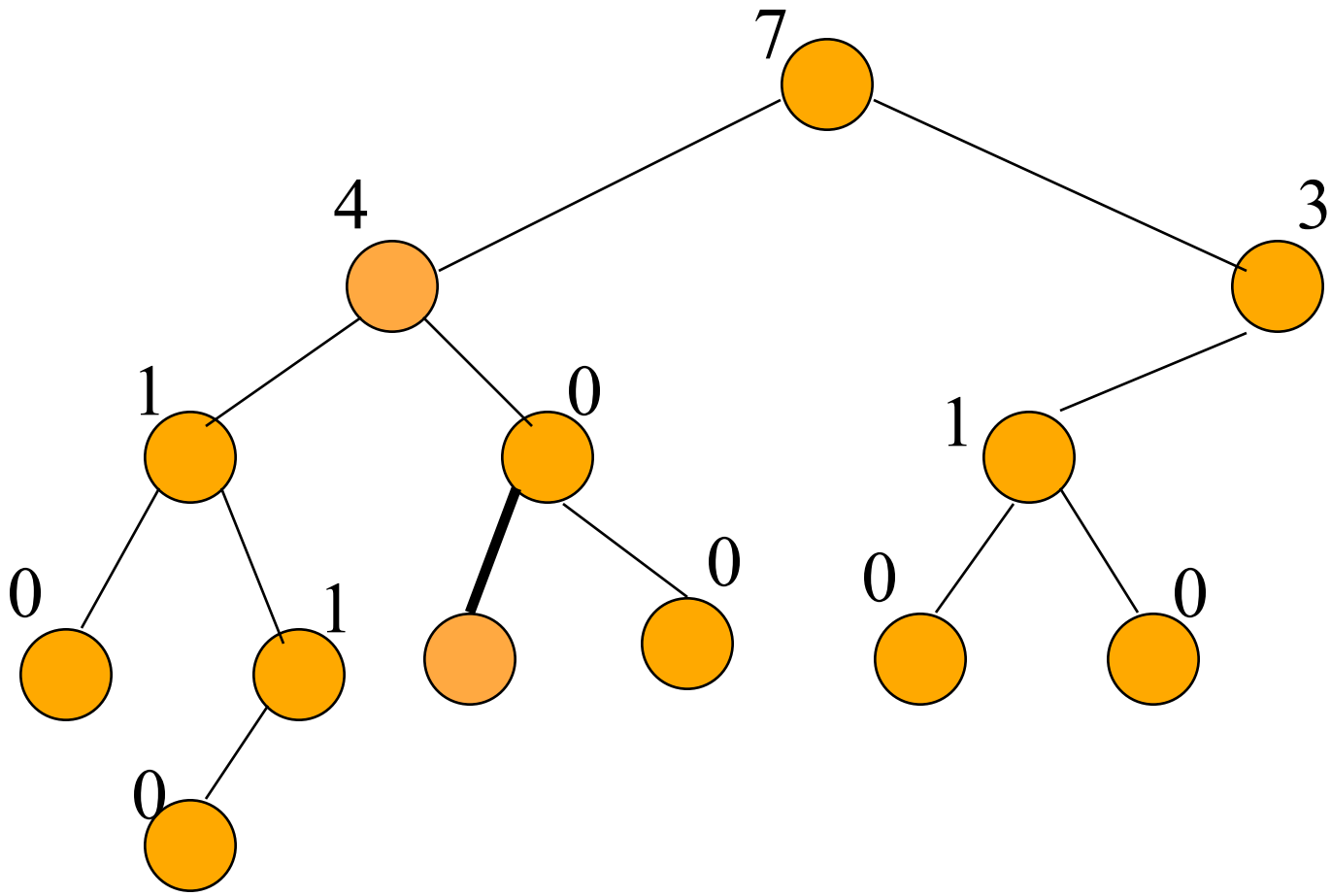
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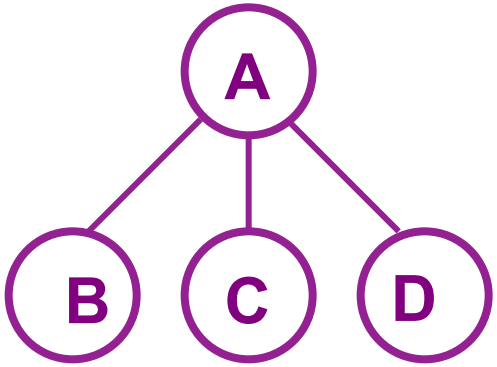
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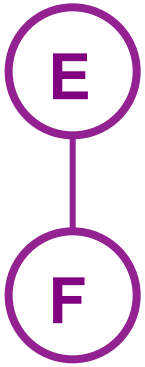
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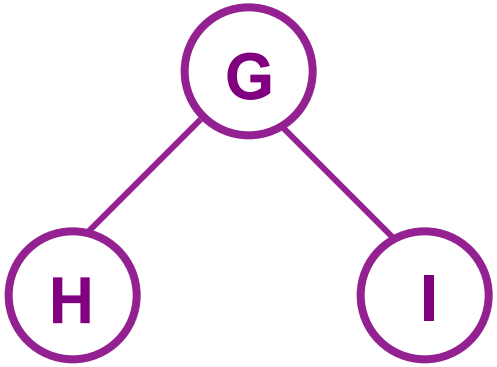
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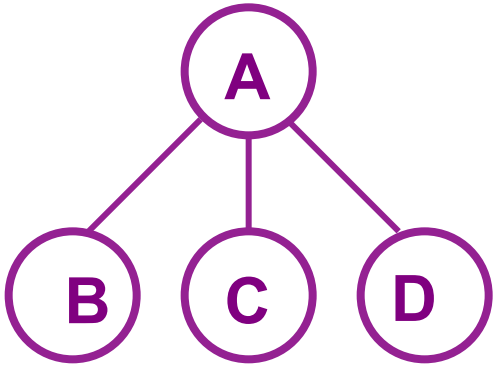
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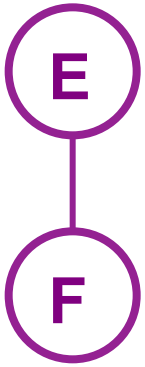
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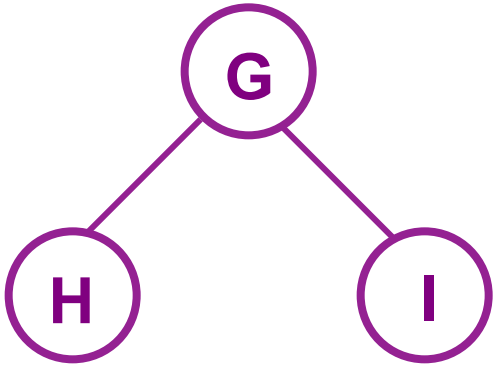
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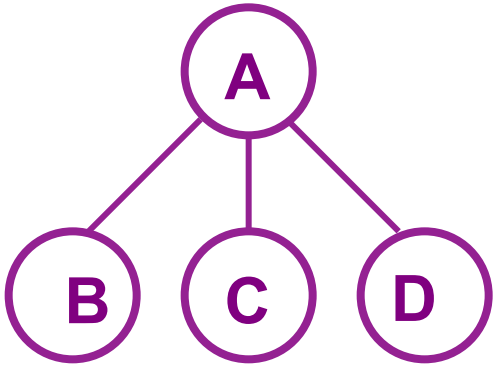
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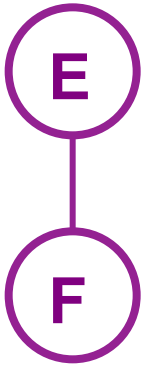
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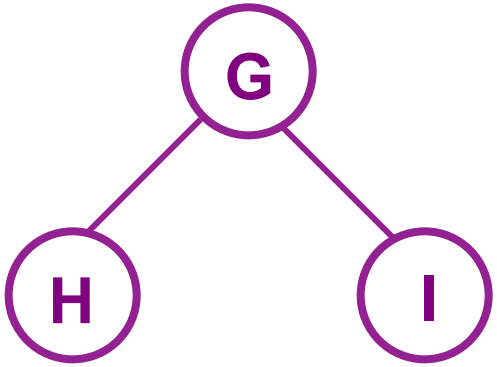
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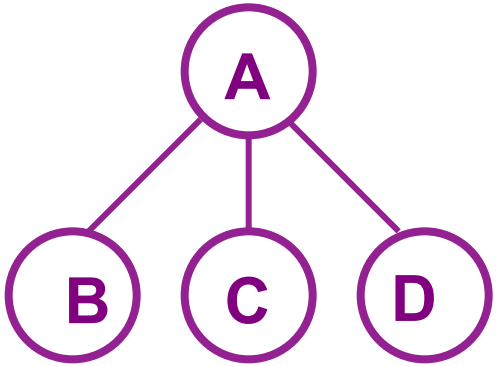
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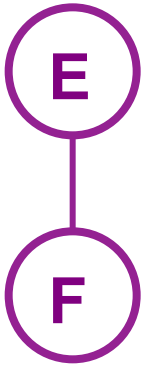
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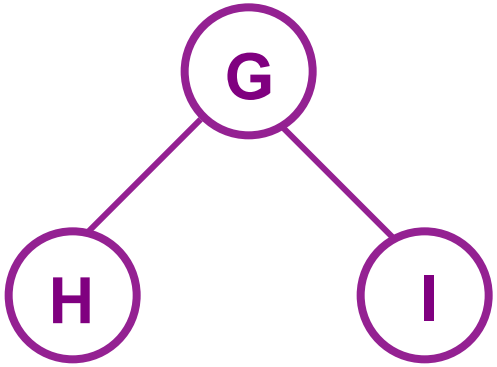
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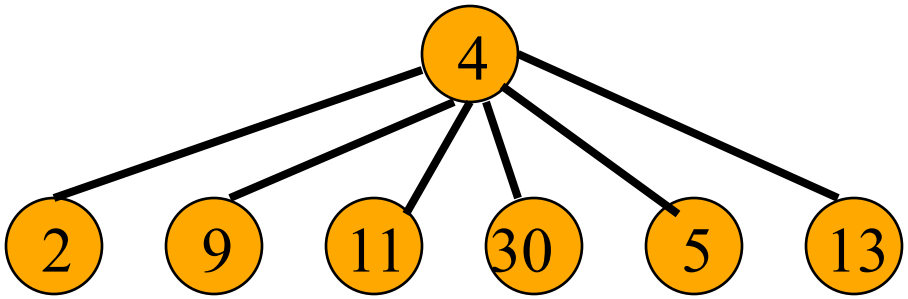

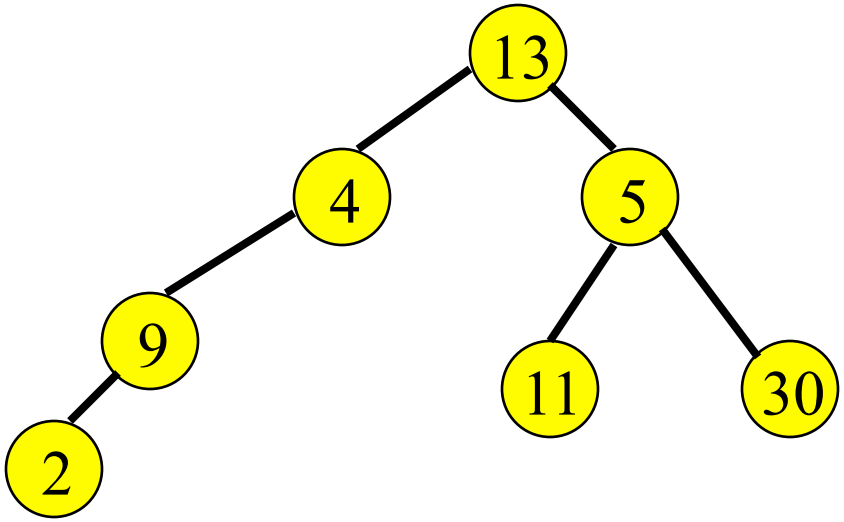

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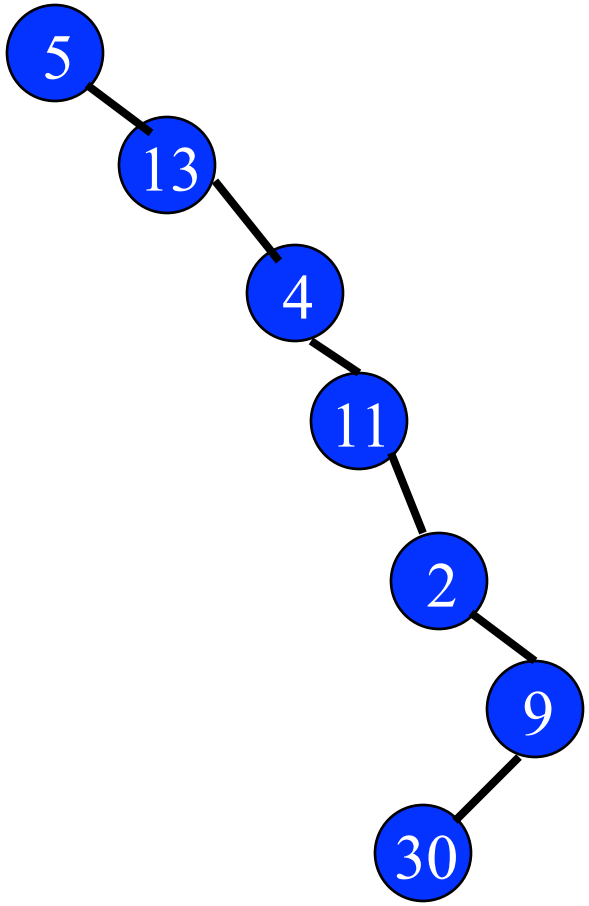

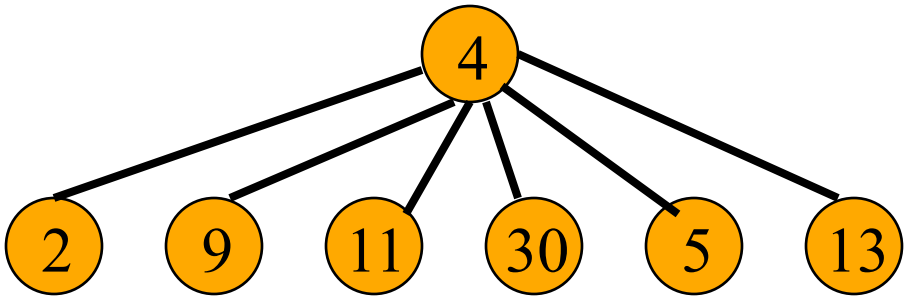
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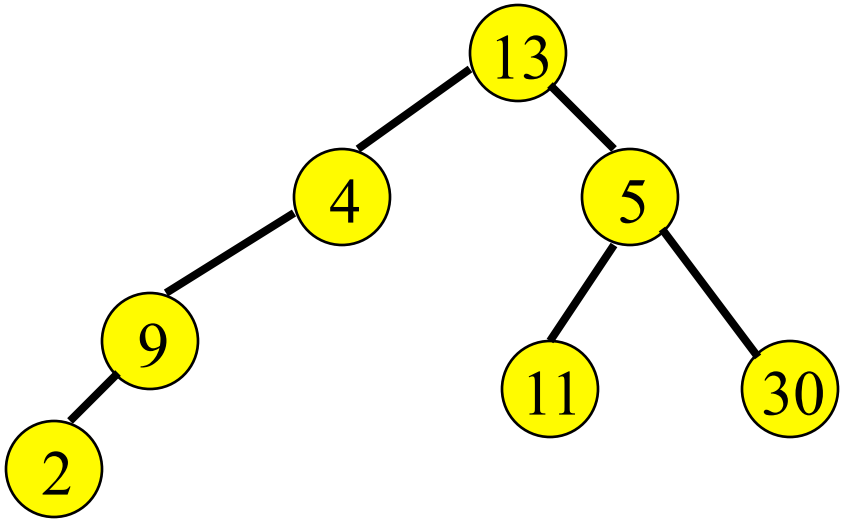
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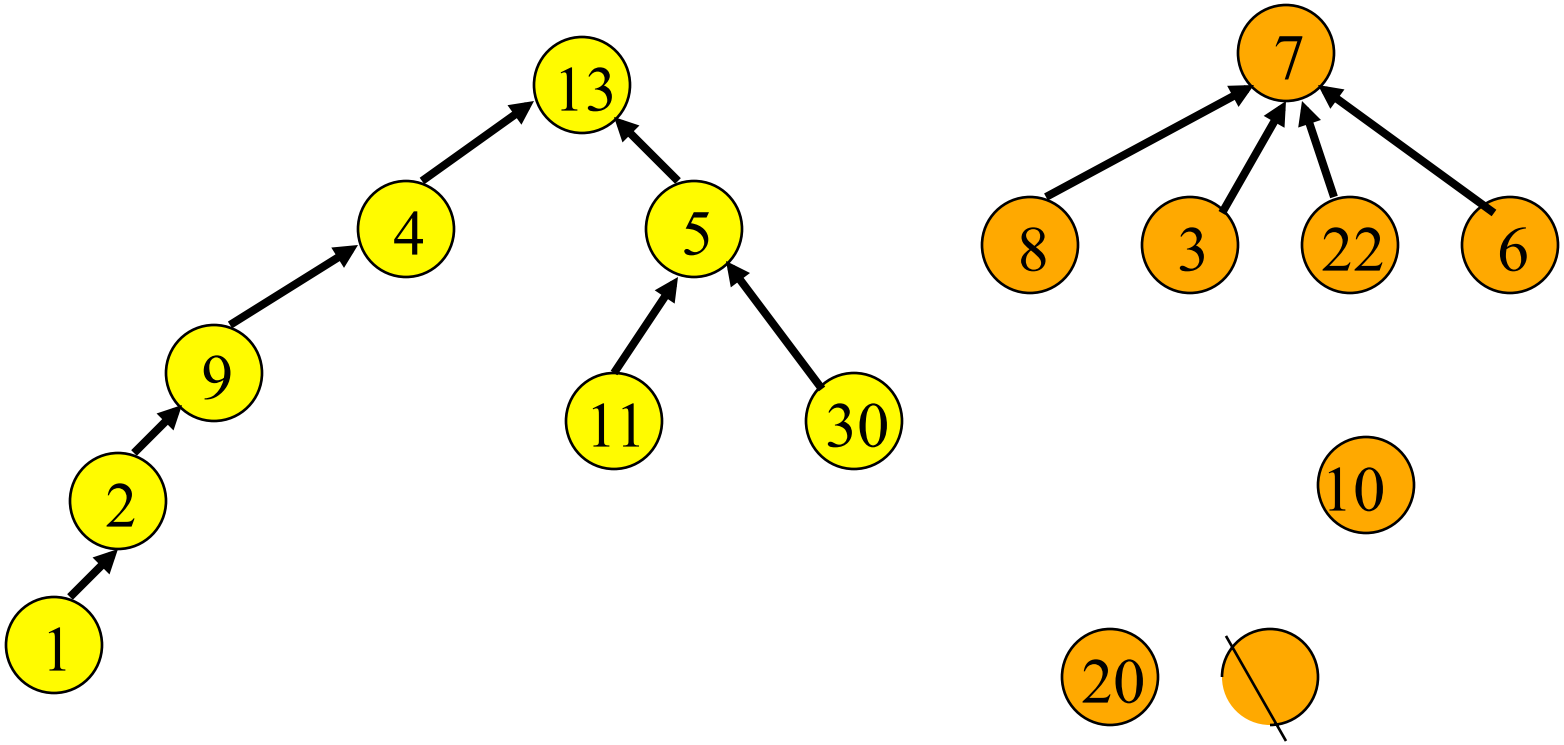
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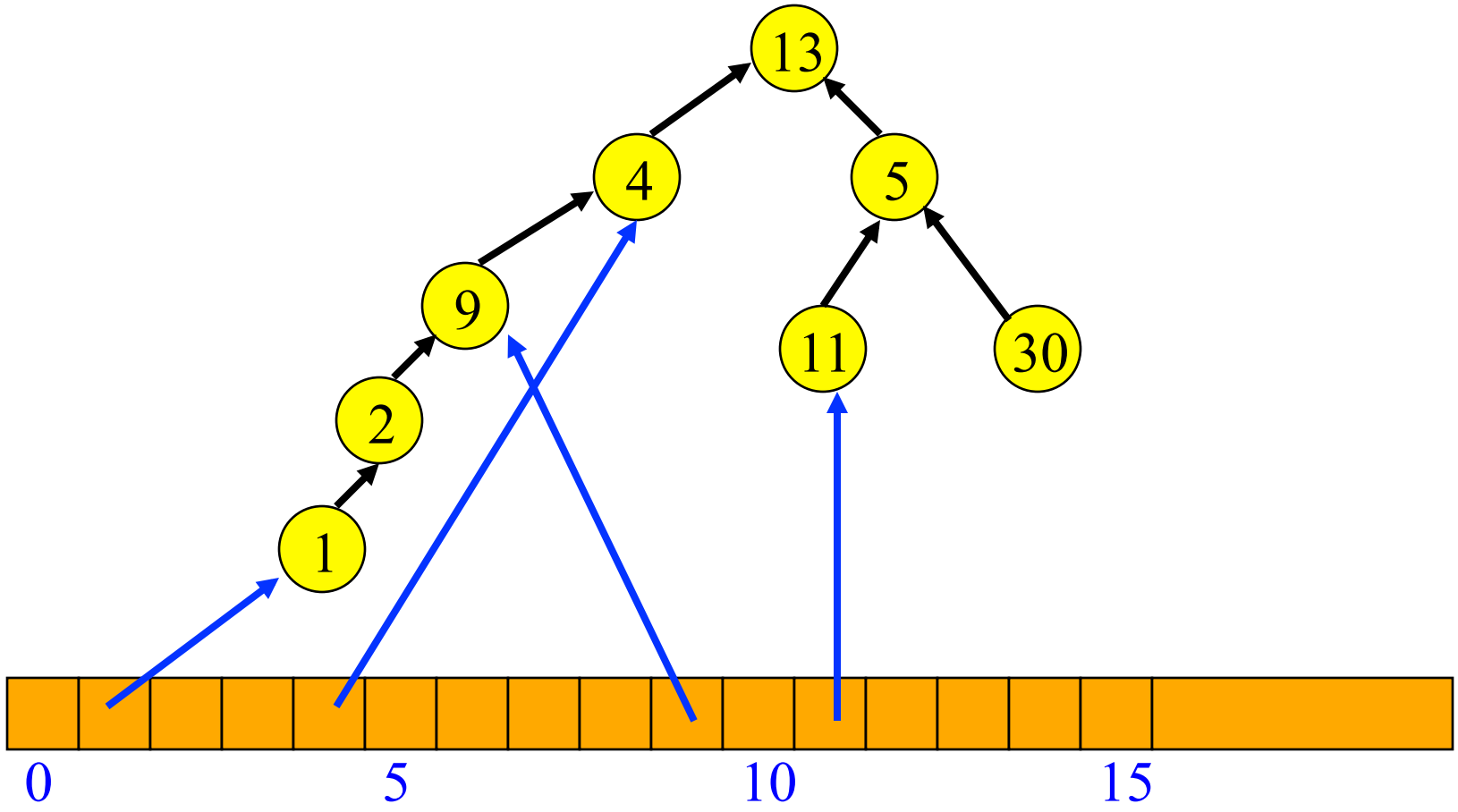
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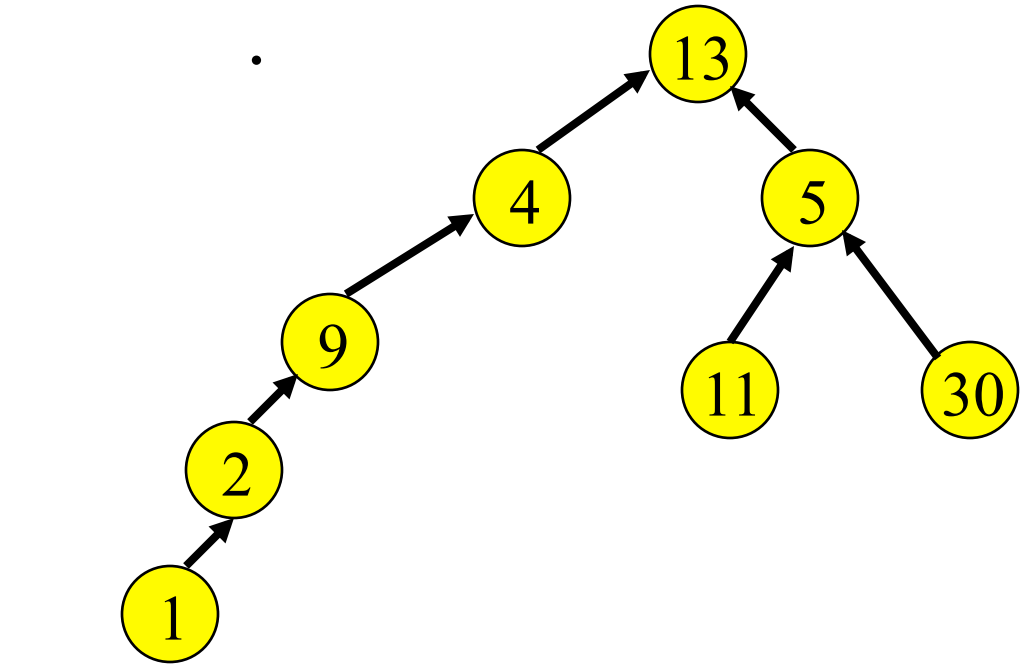
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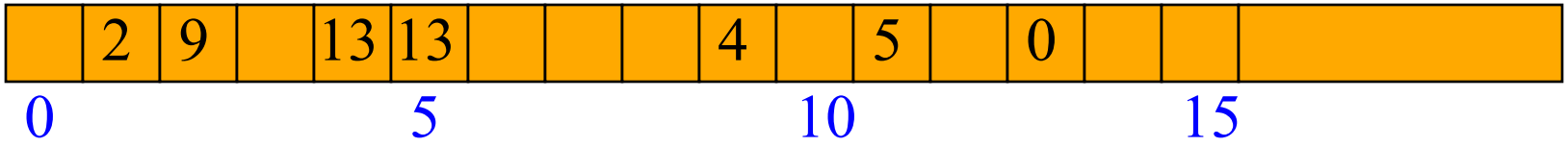
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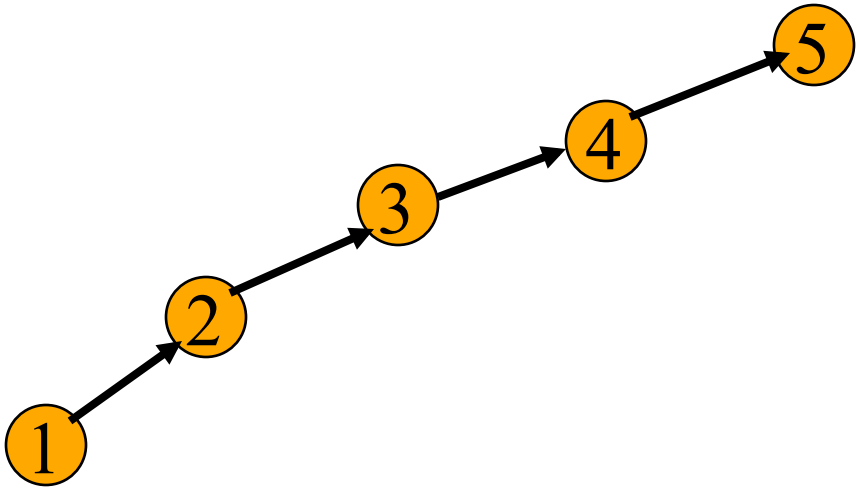

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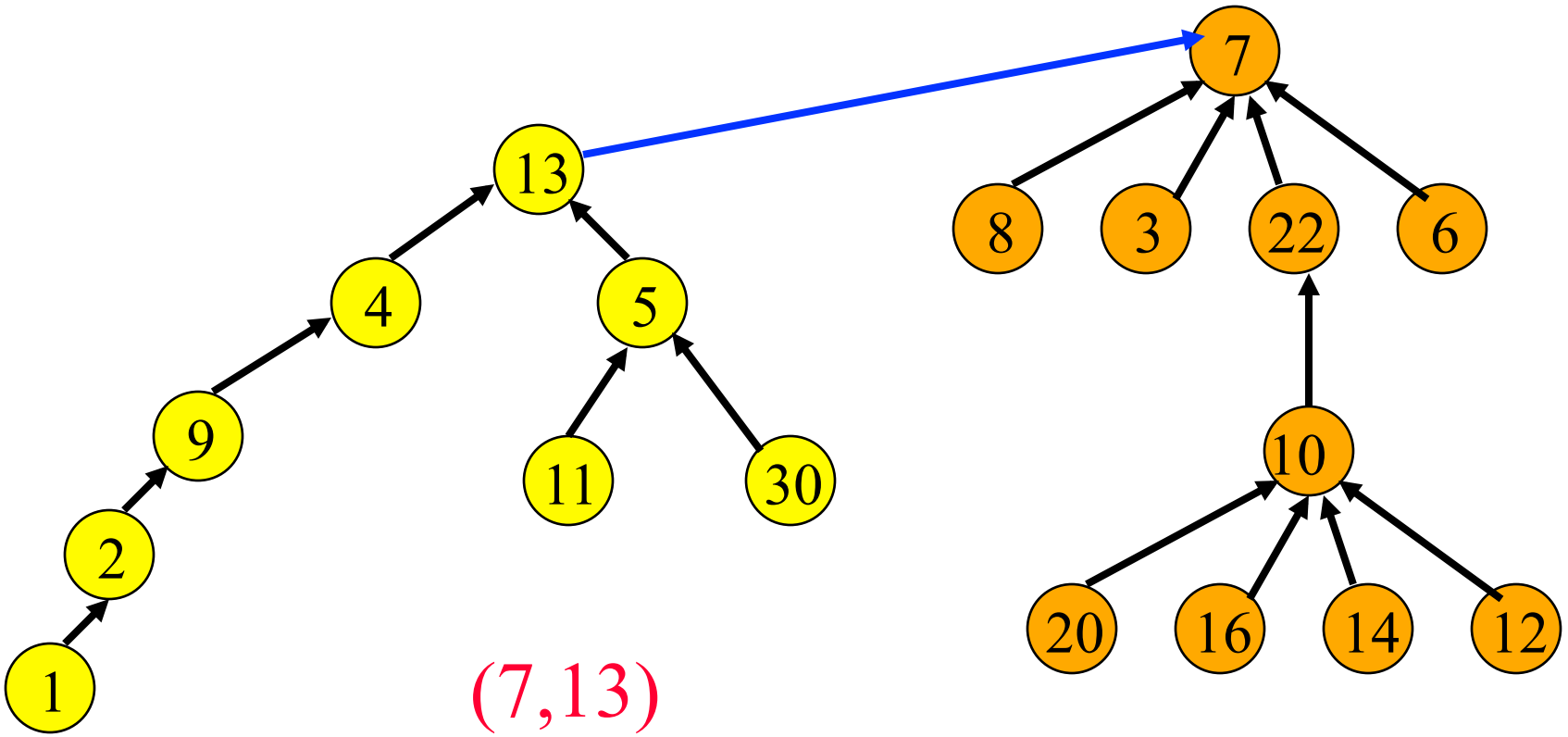
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Lemma 5.5

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- **Lemma 5.5**

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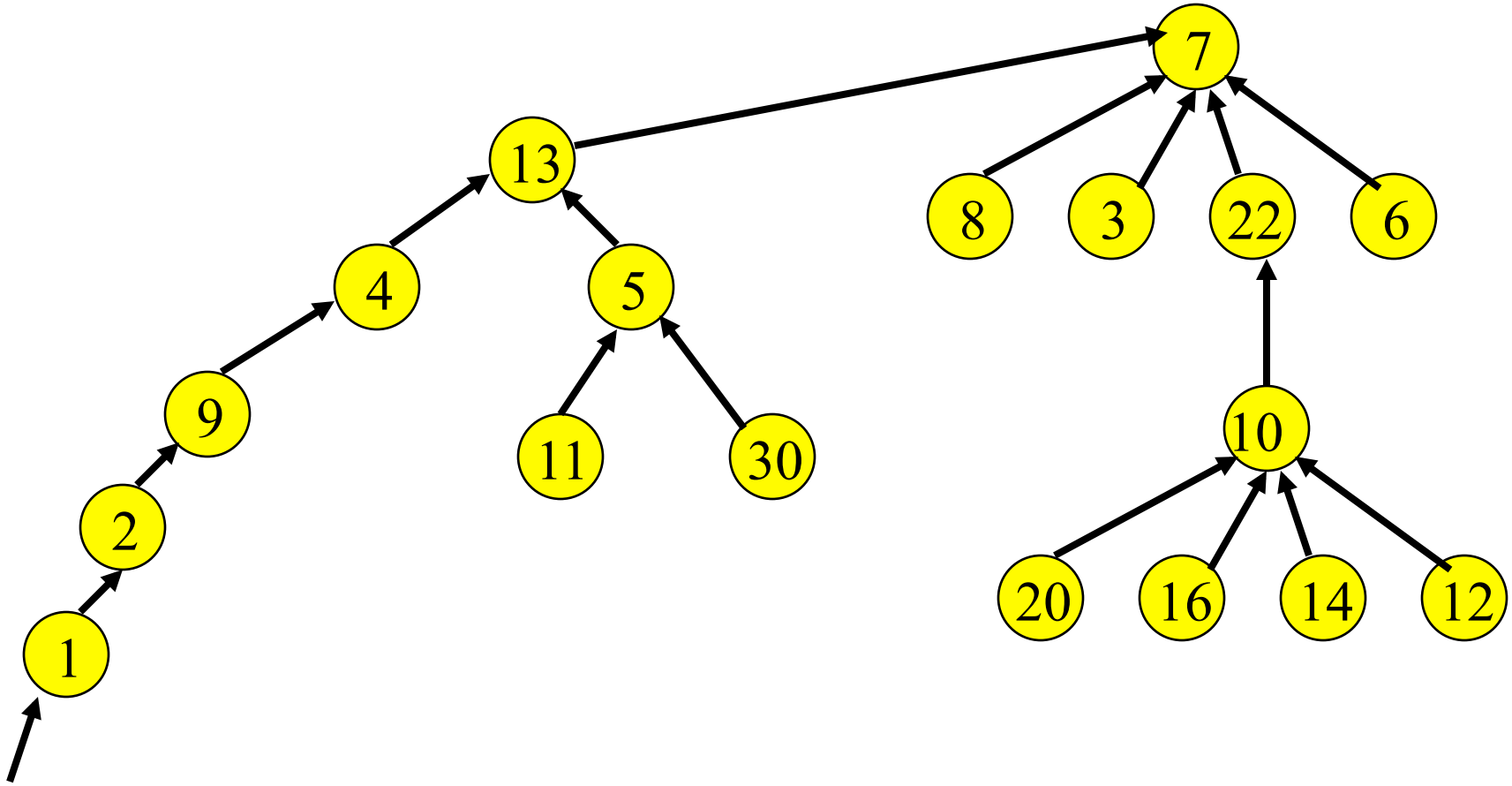

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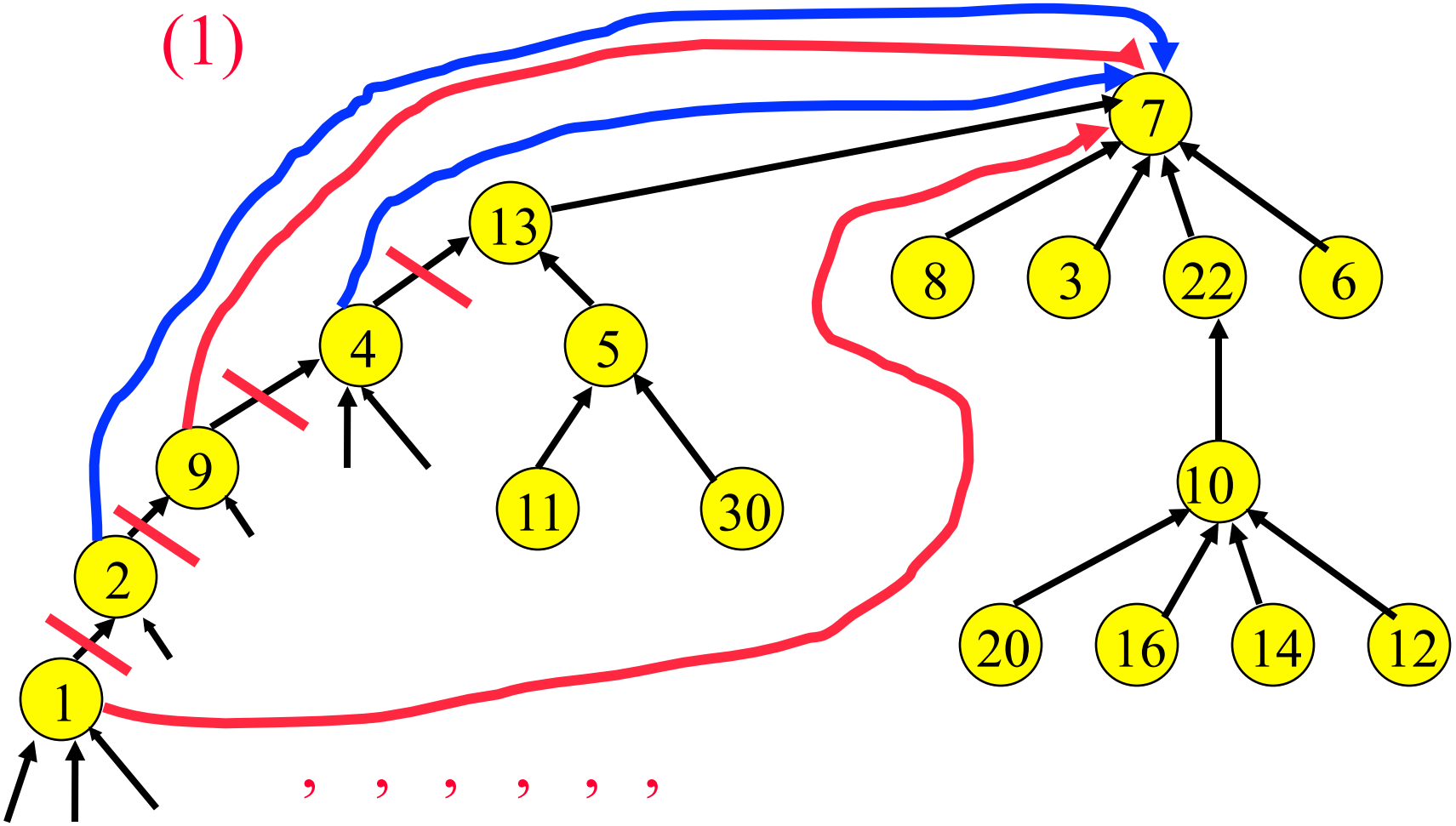

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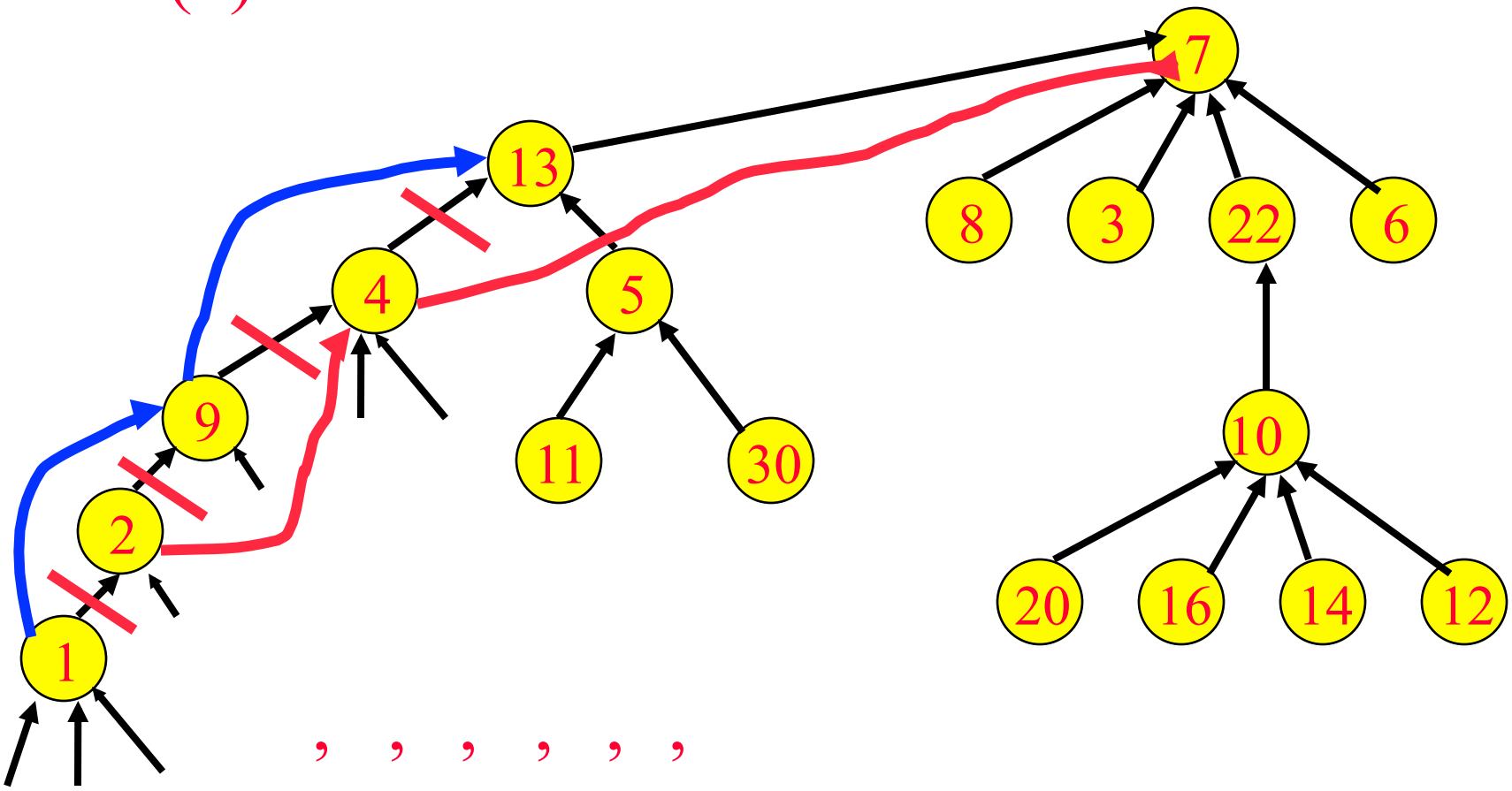

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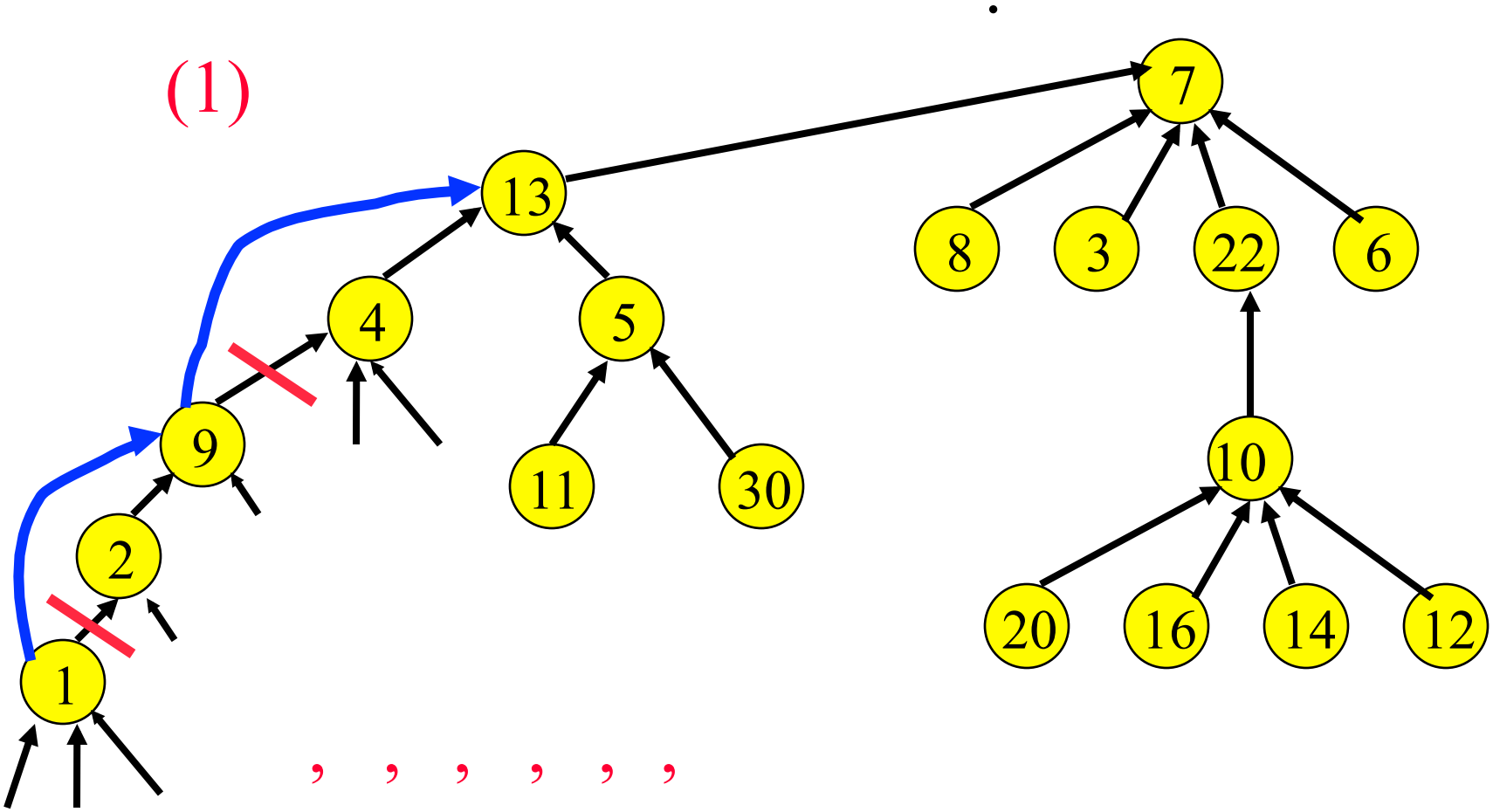
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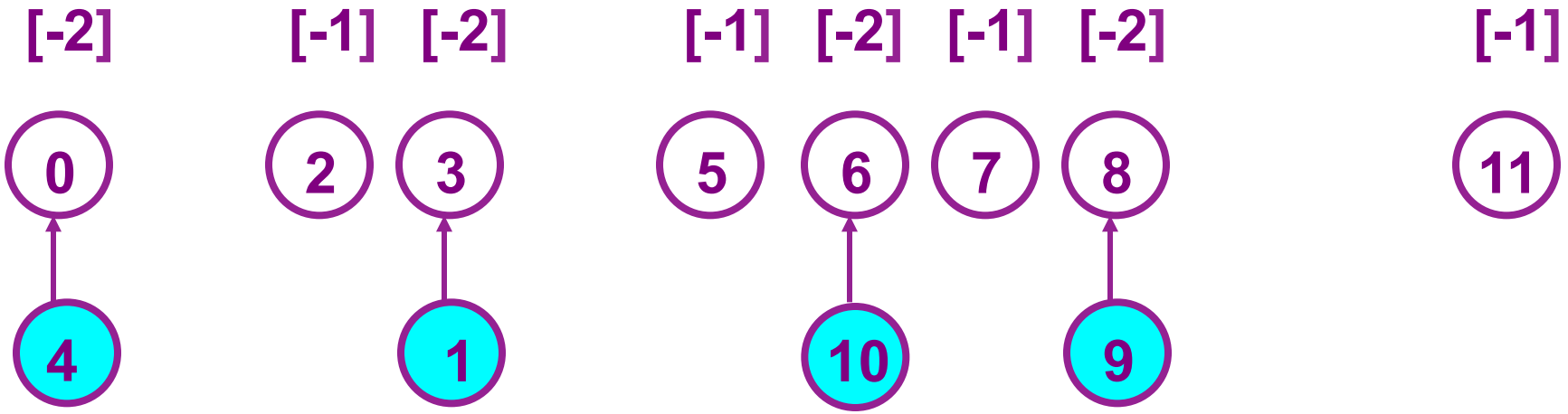
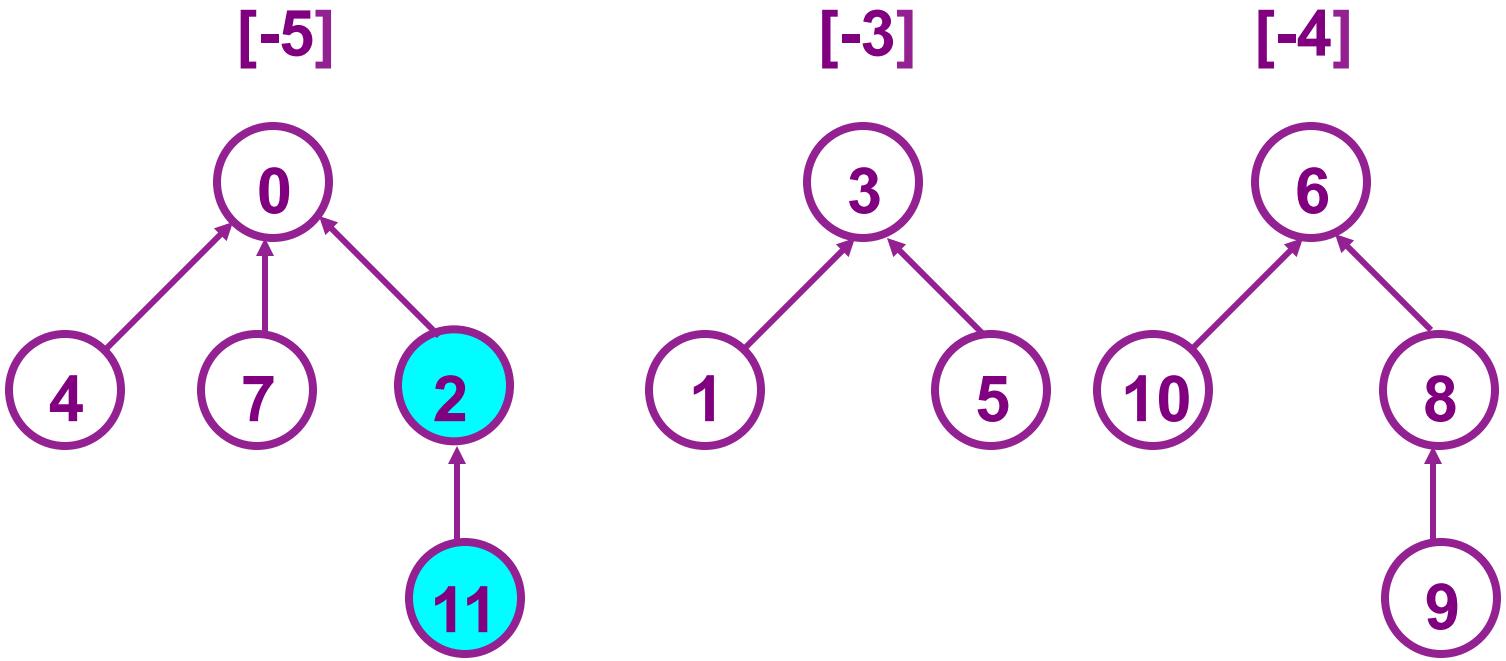
(a) Initial trees



(b) After processing 0 ≡ 4, 3 ≡ 1, 6 ≡10, and 8 ≡ 9



(d) After processing 11 ≡ 0

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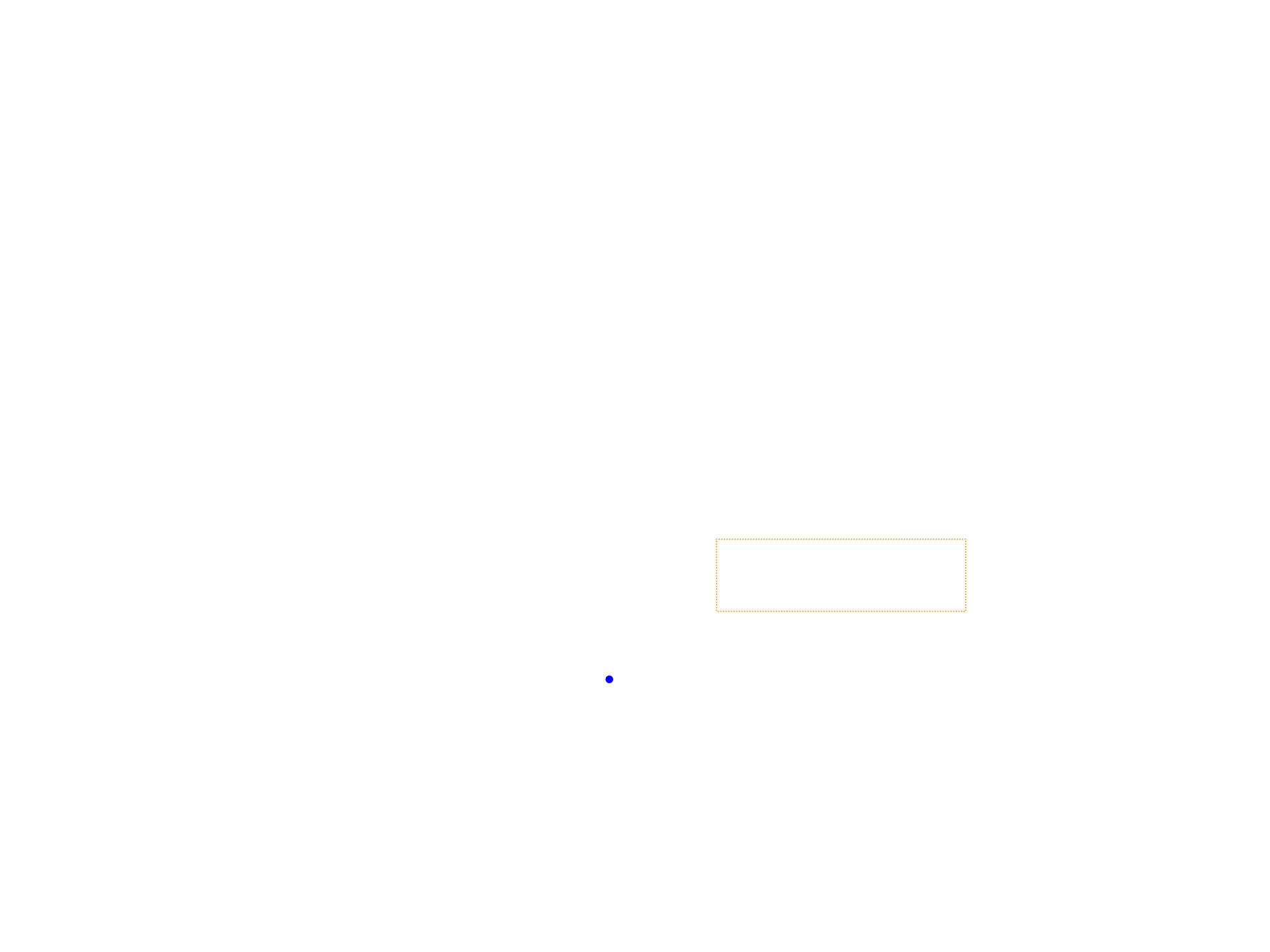

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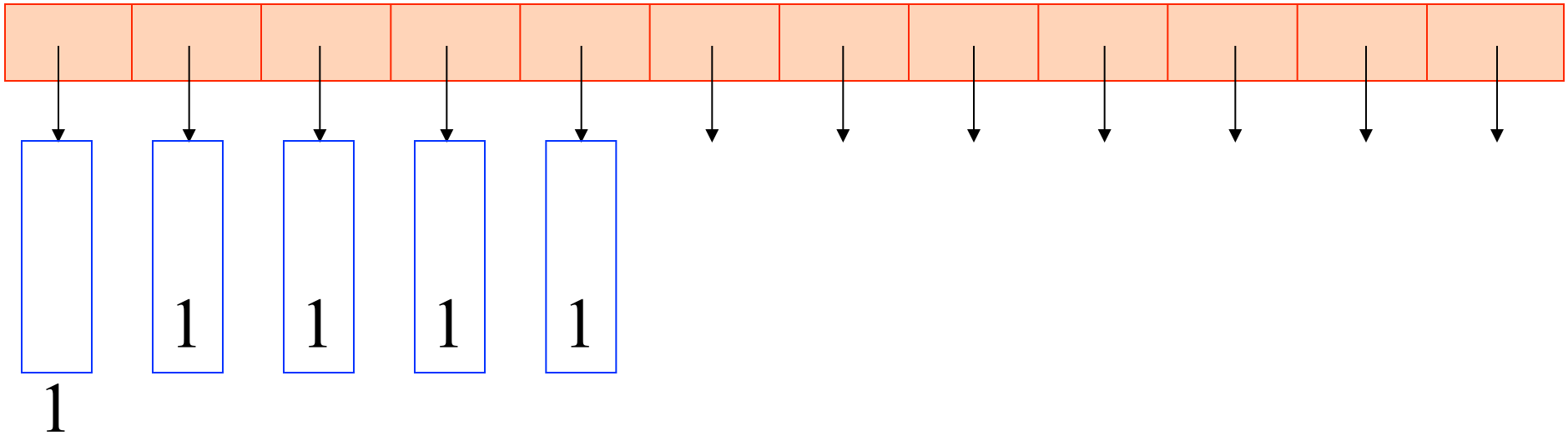
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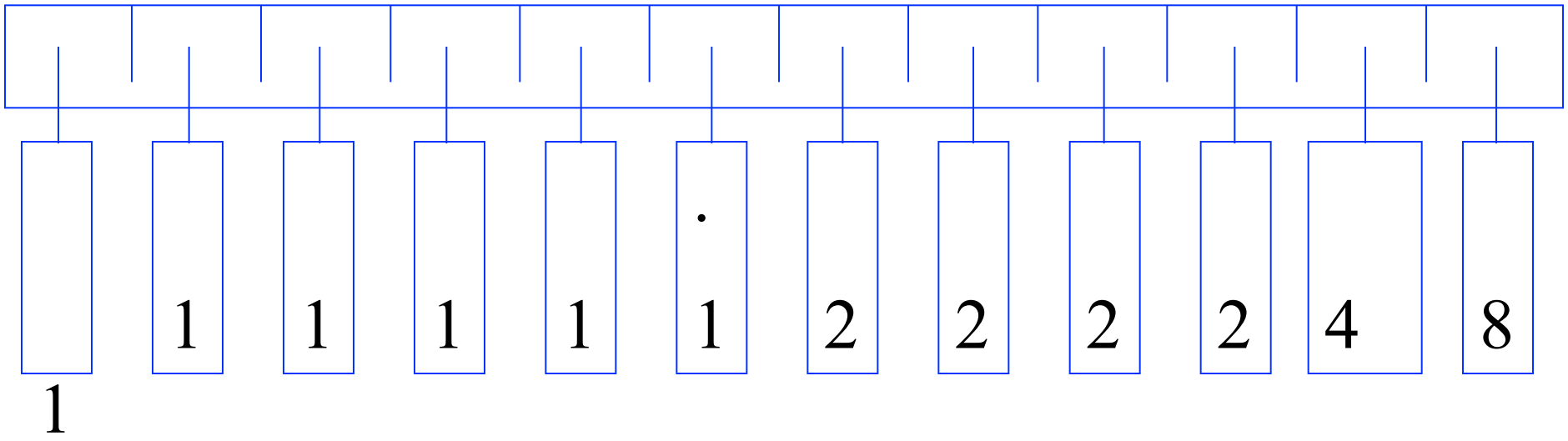
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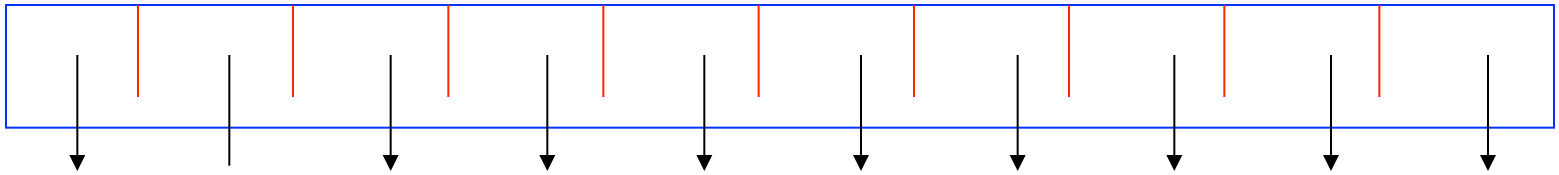

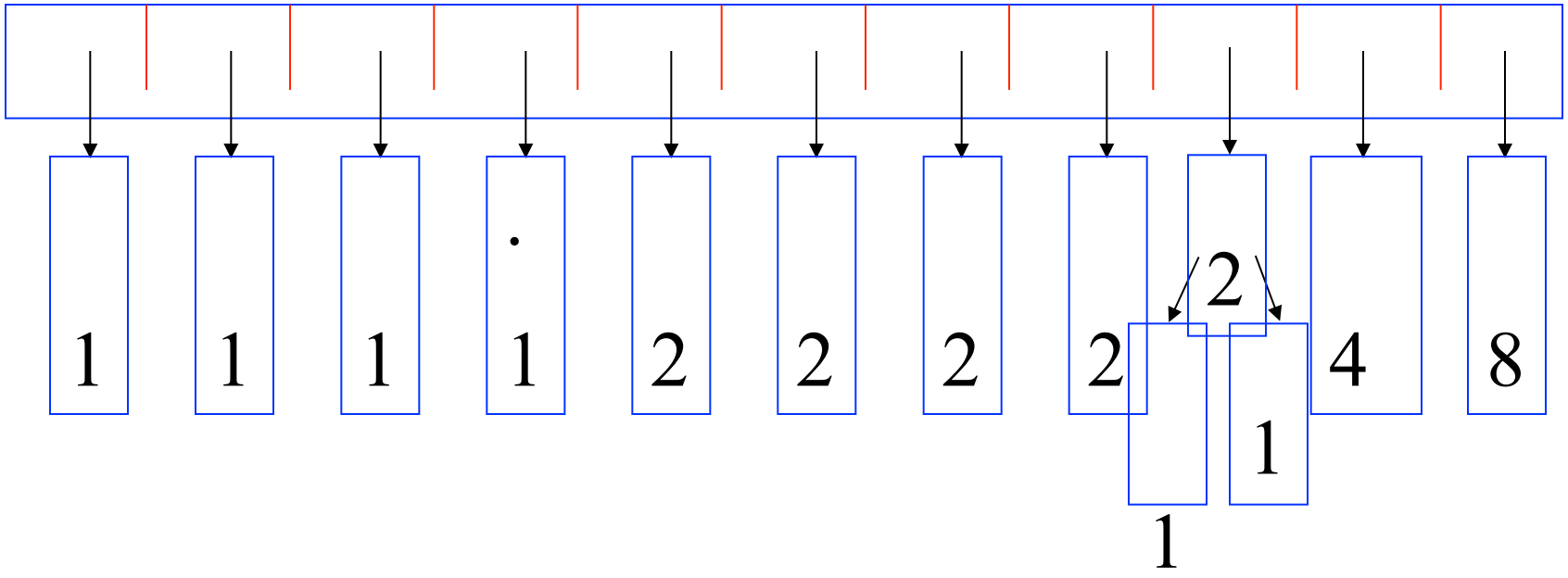
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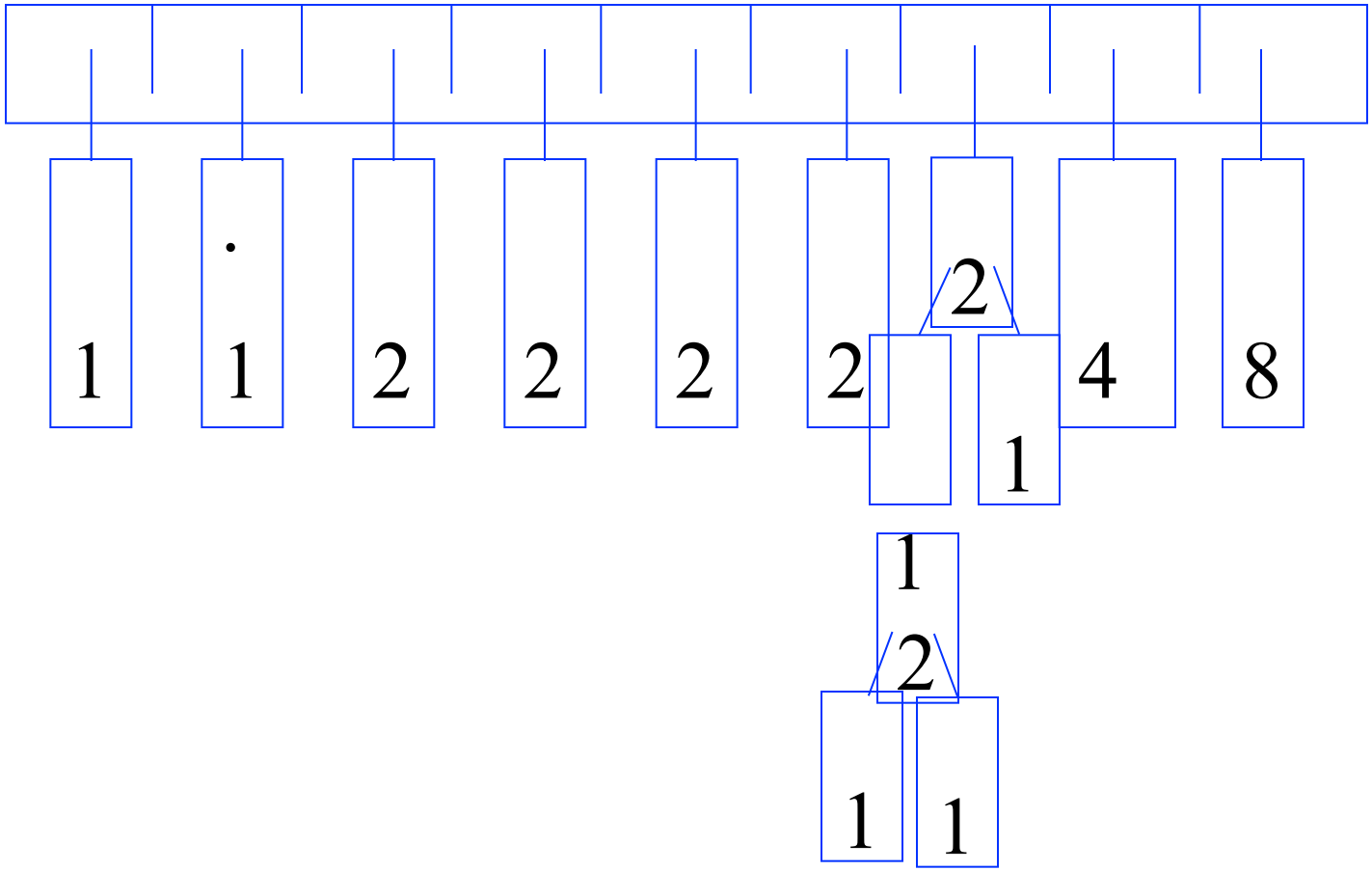
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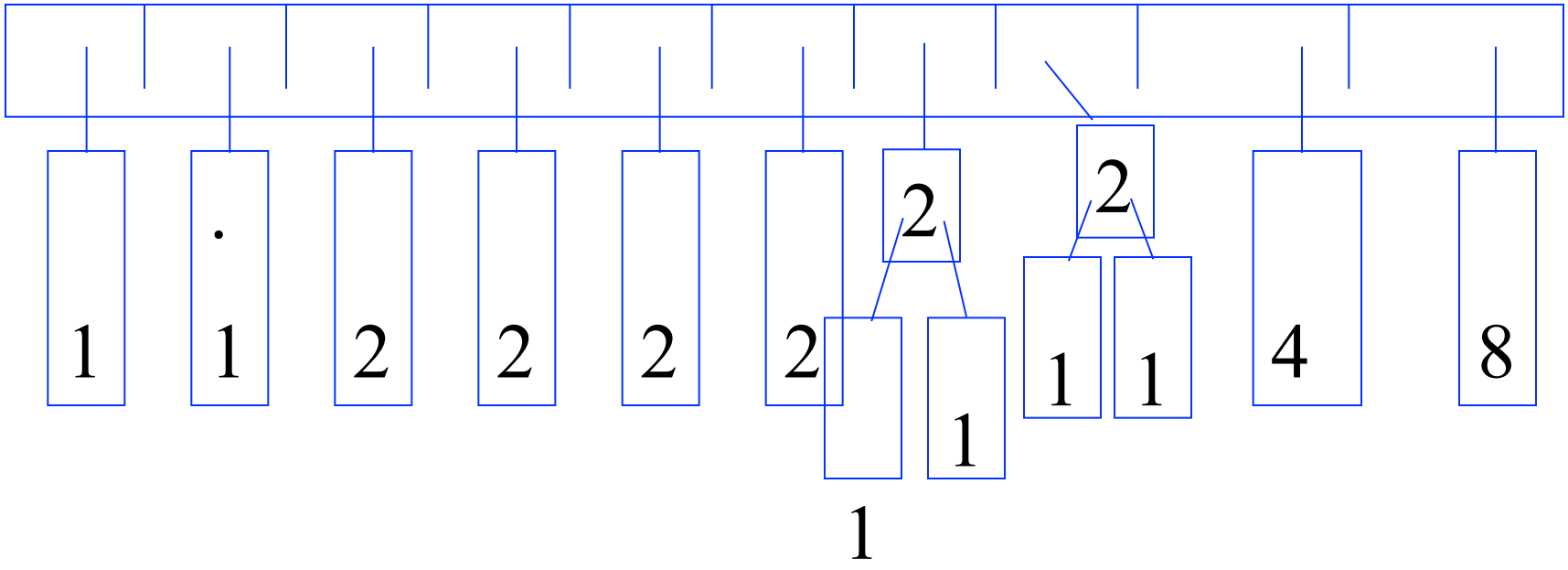
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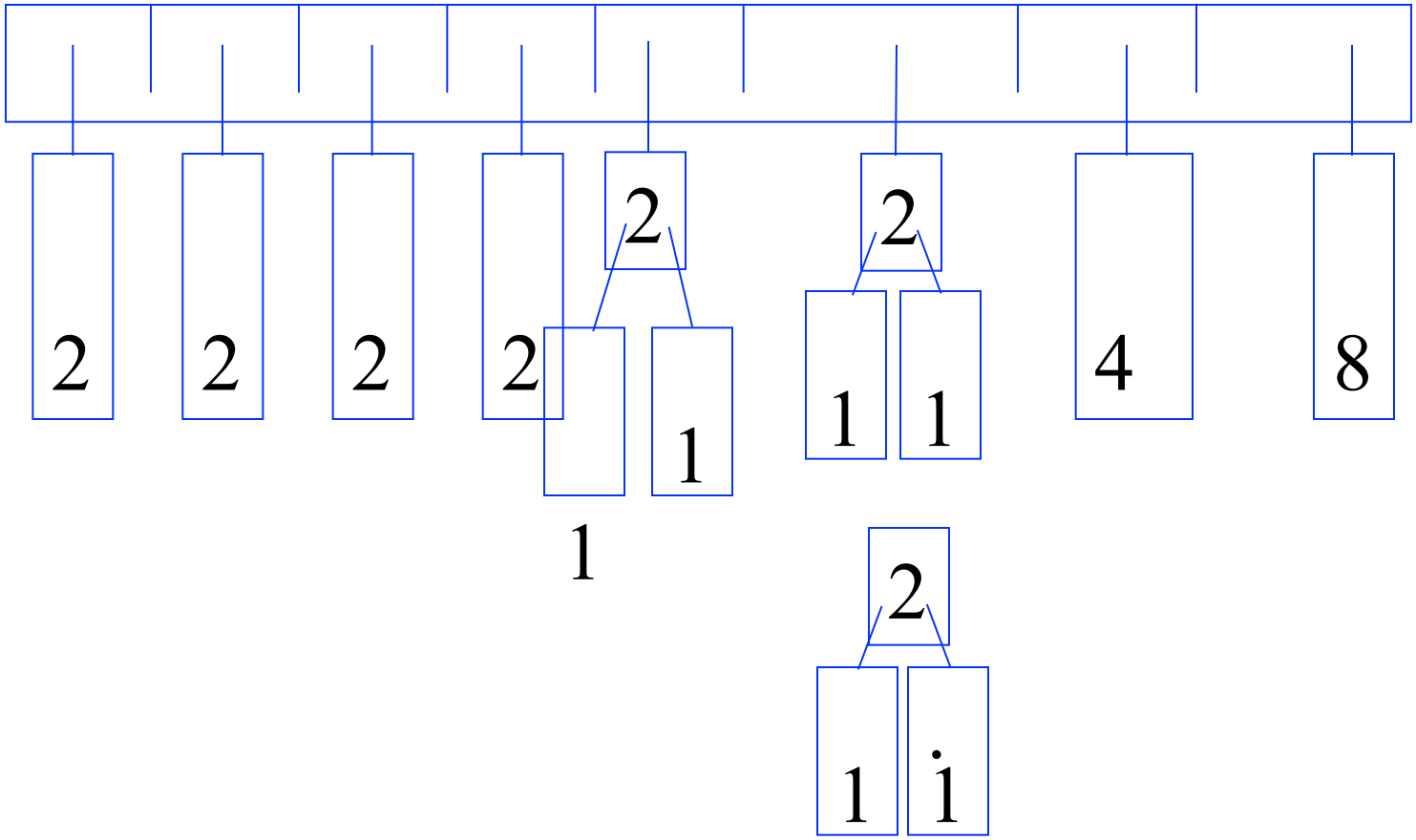
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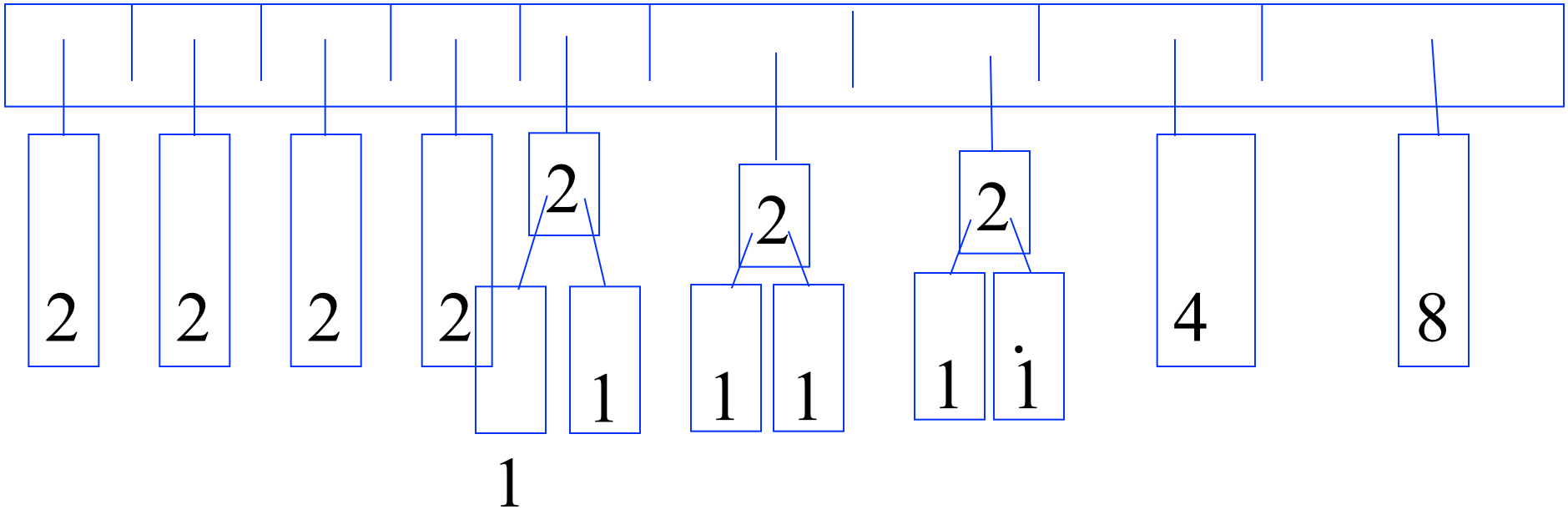
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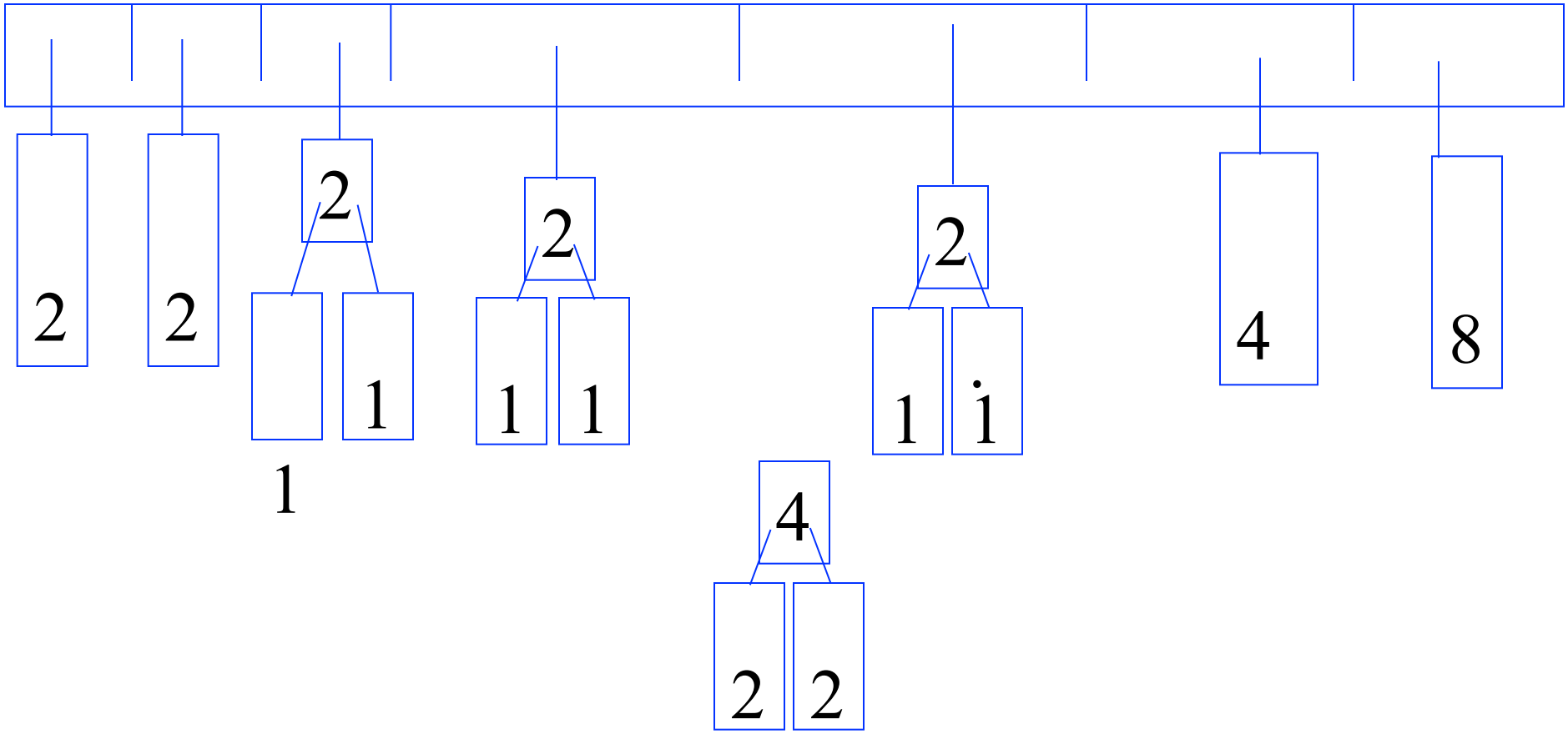
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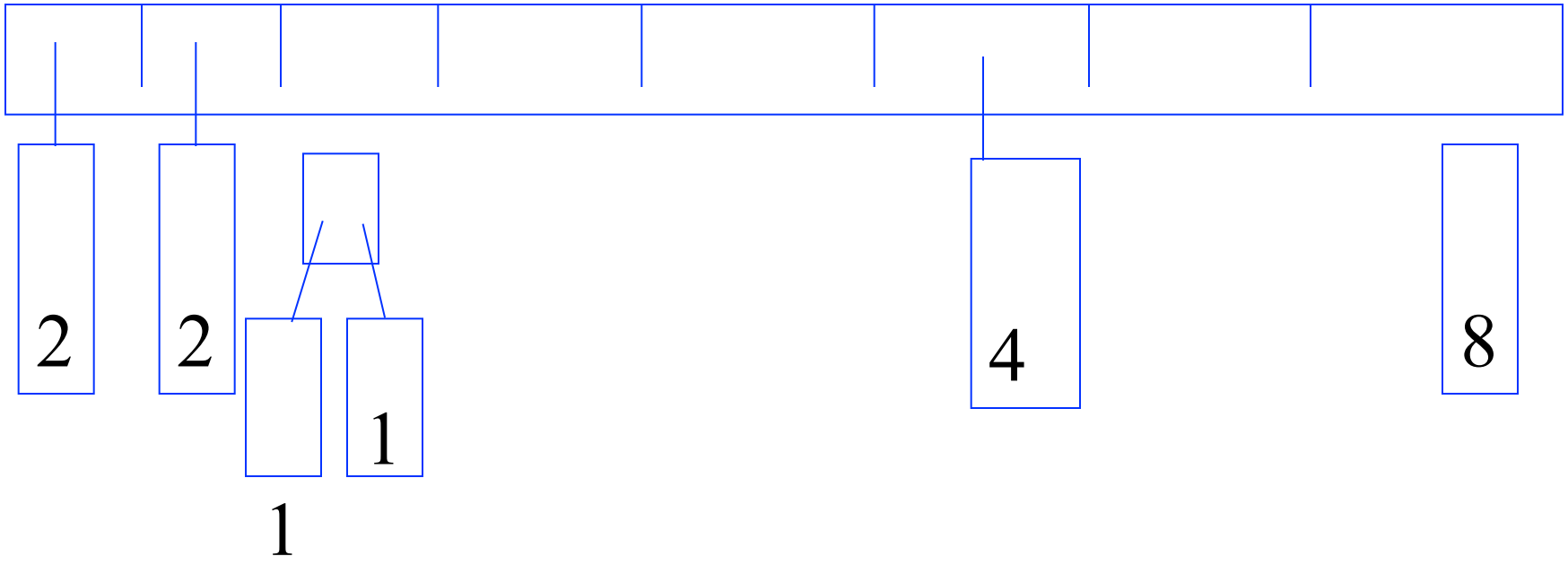
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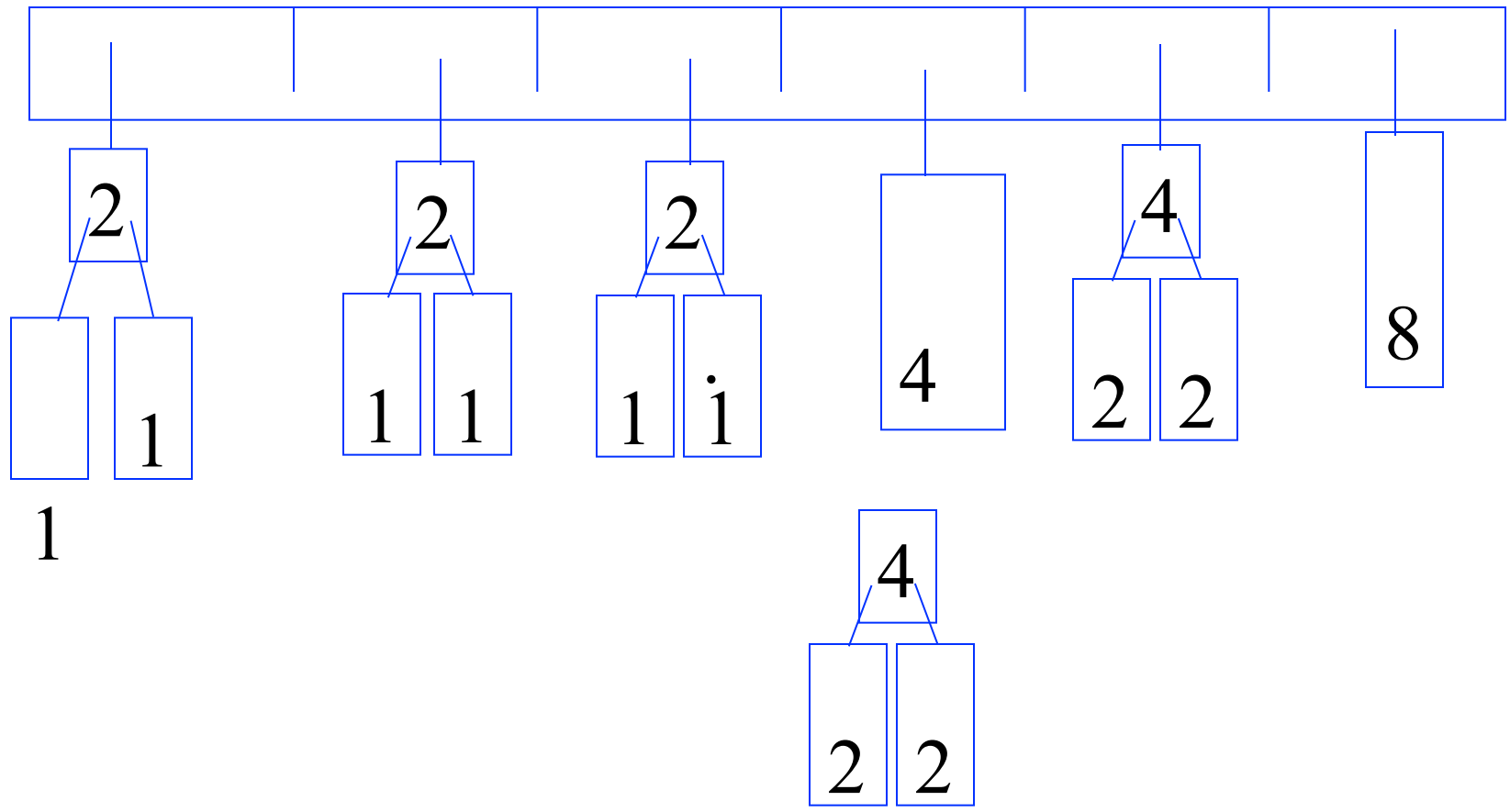
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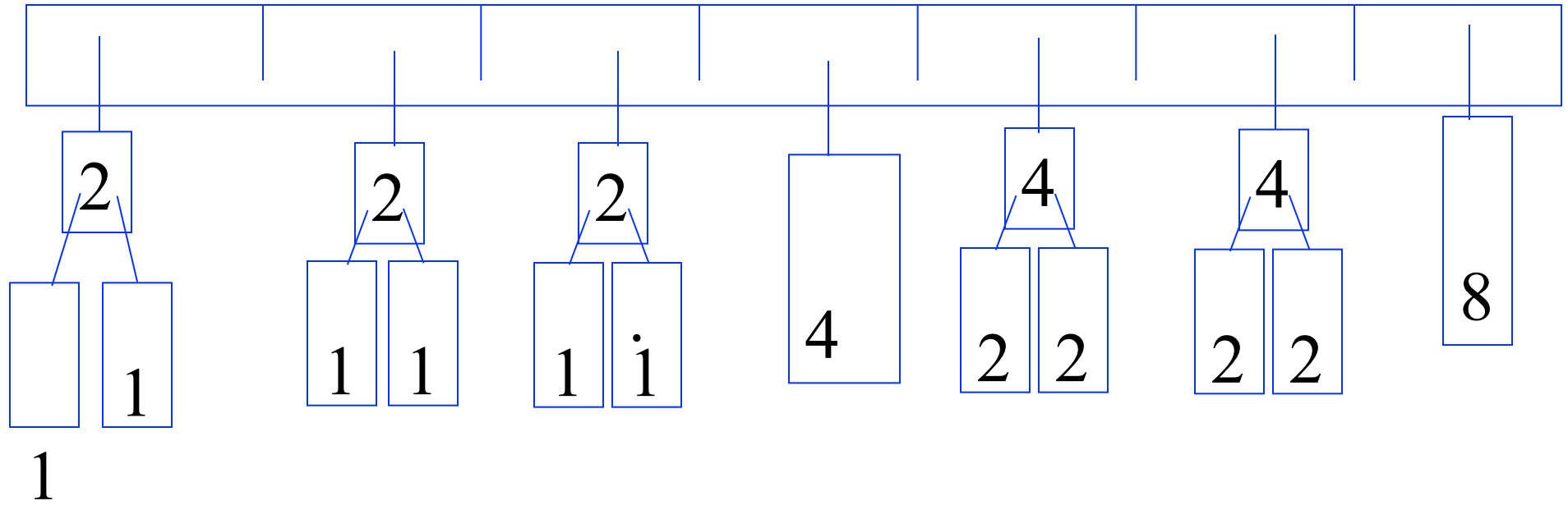
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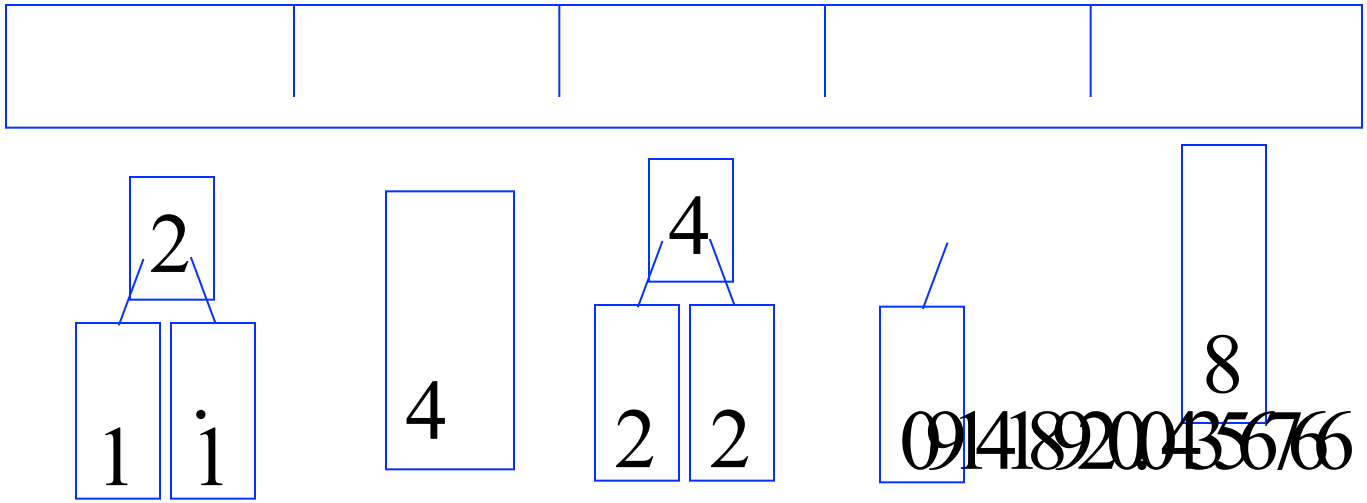
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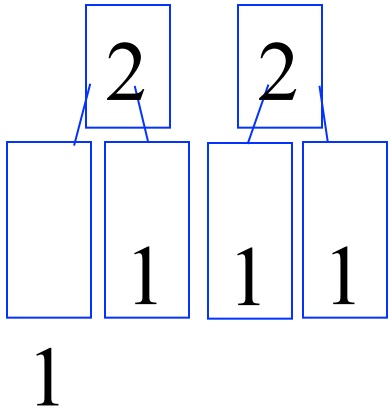
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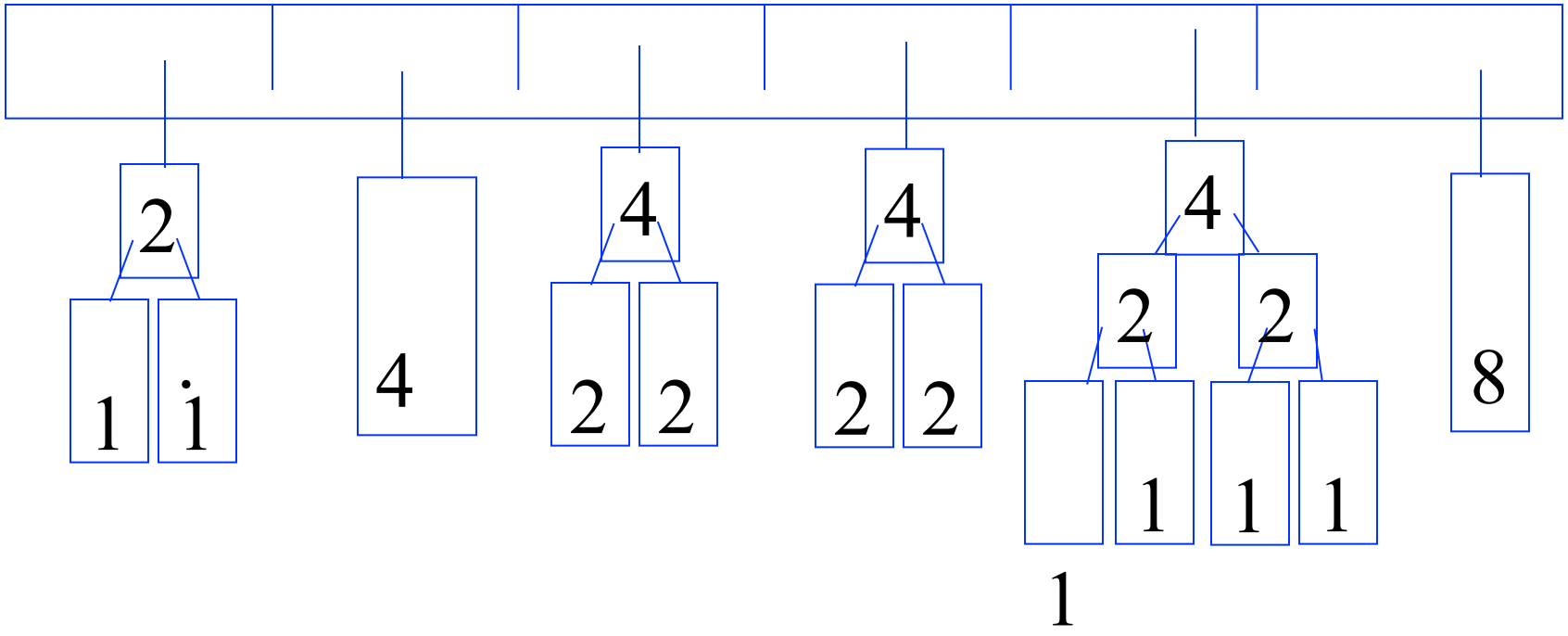
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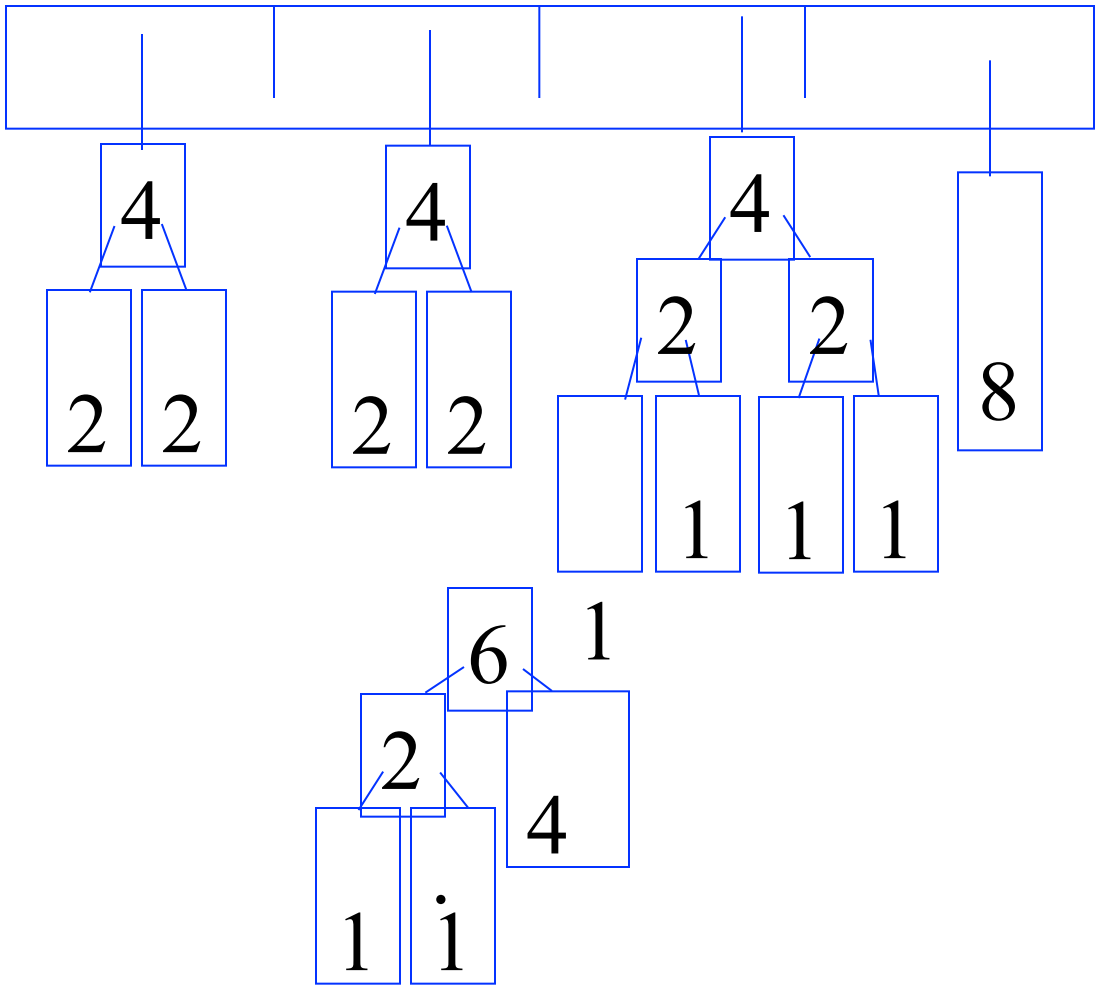
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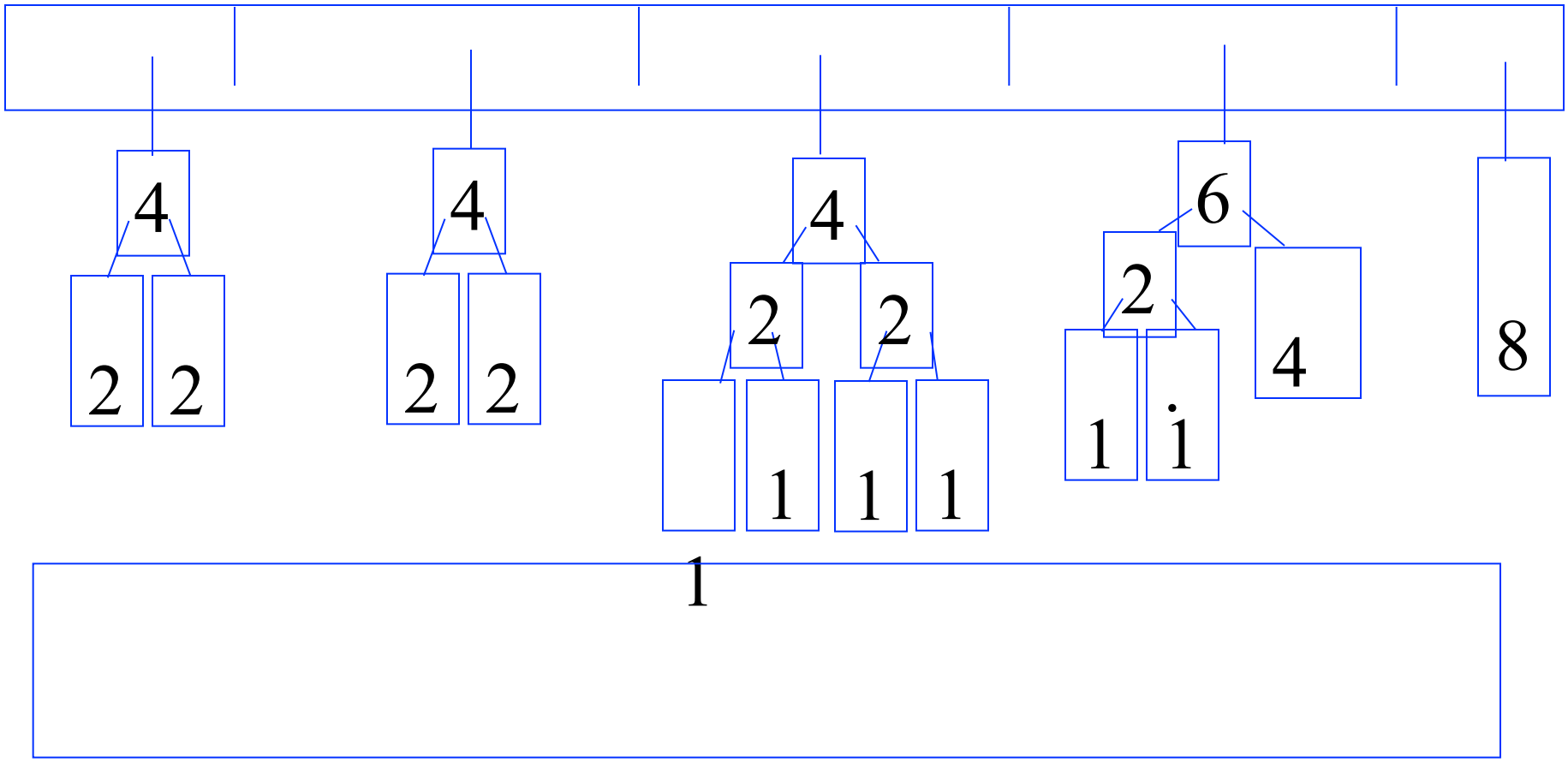
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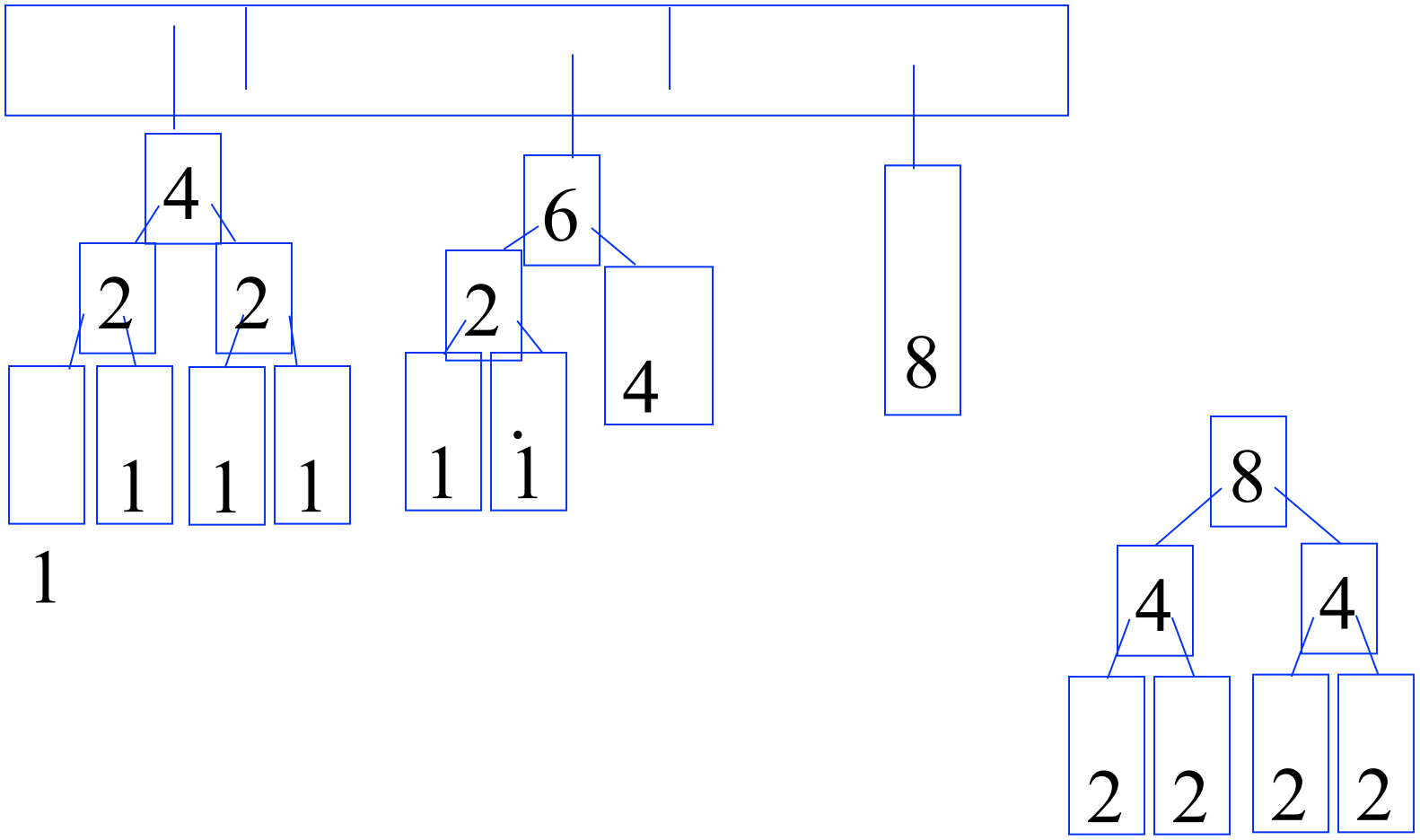
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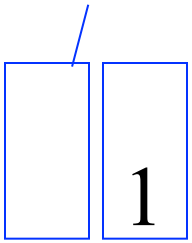

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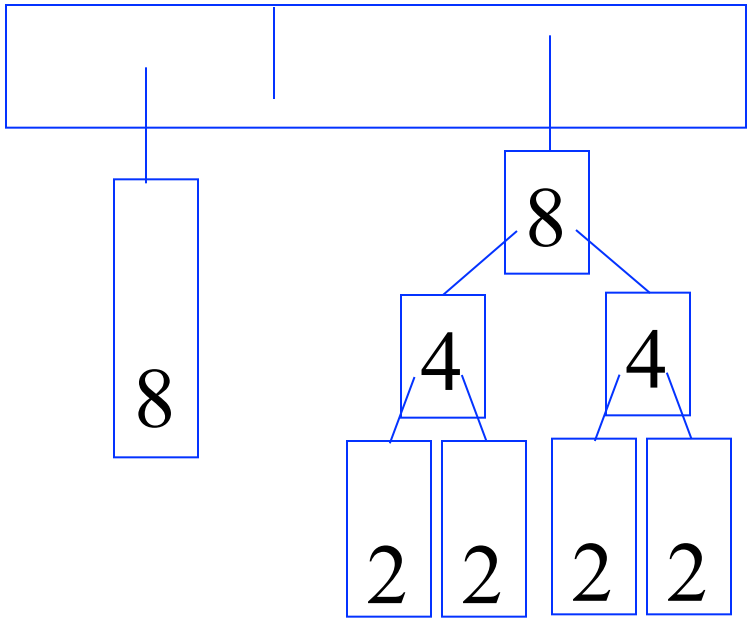
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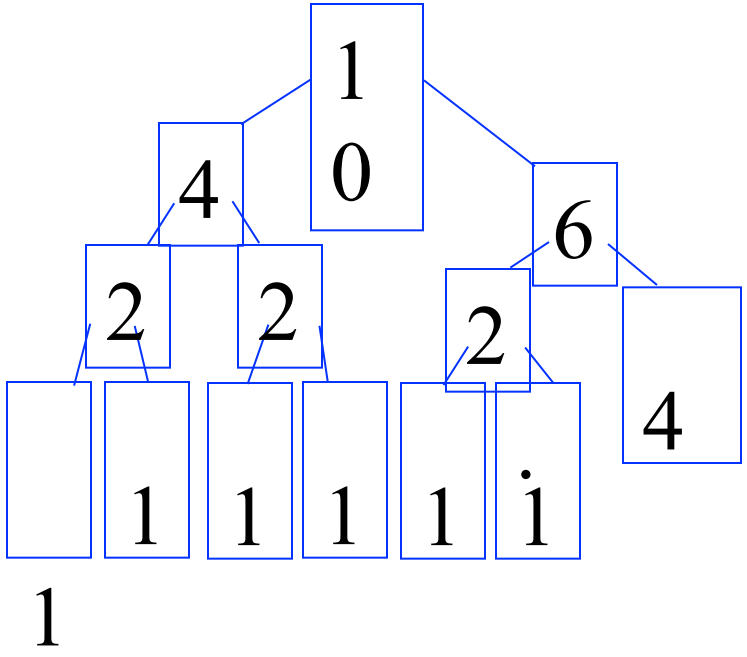
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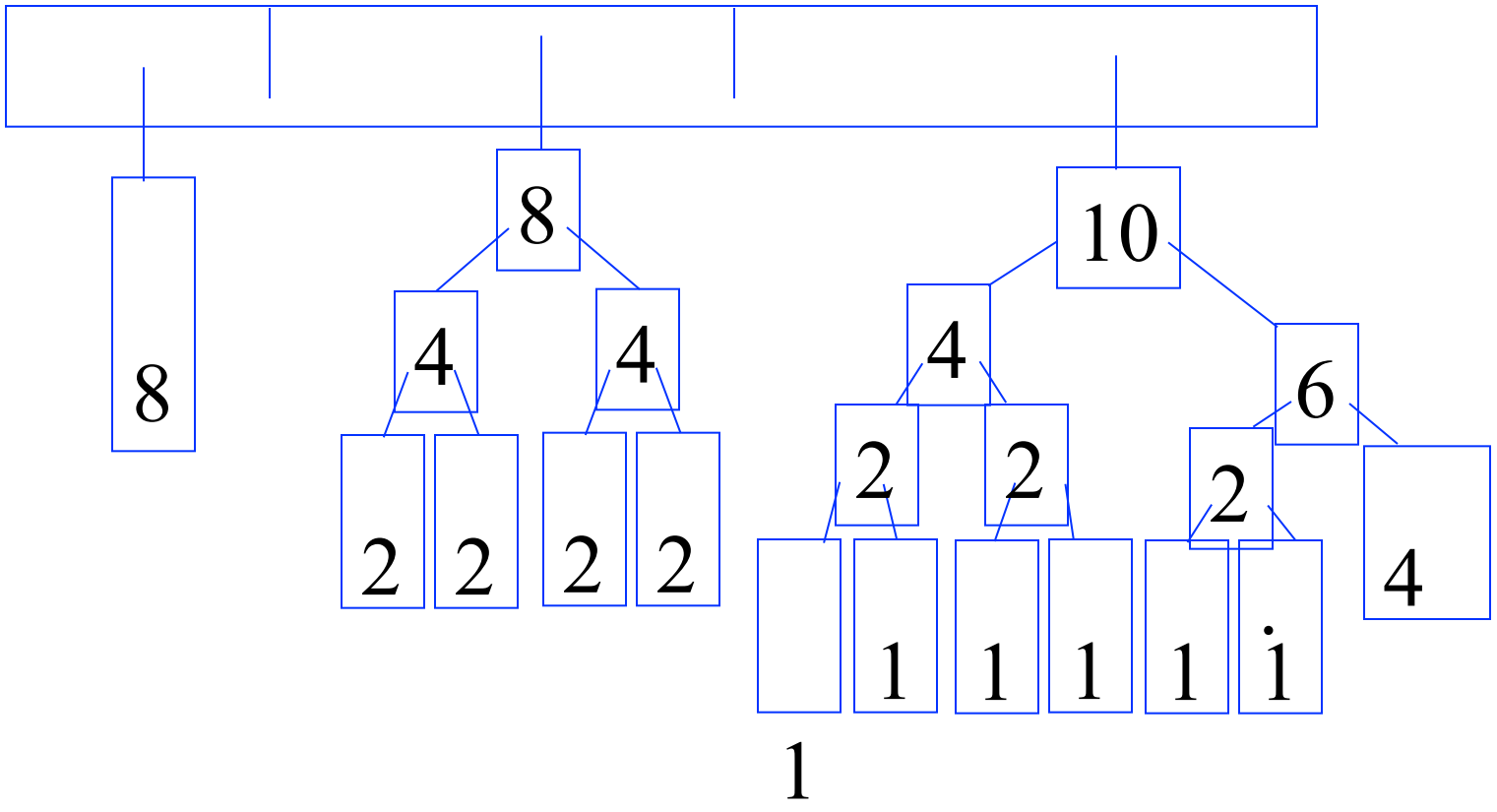
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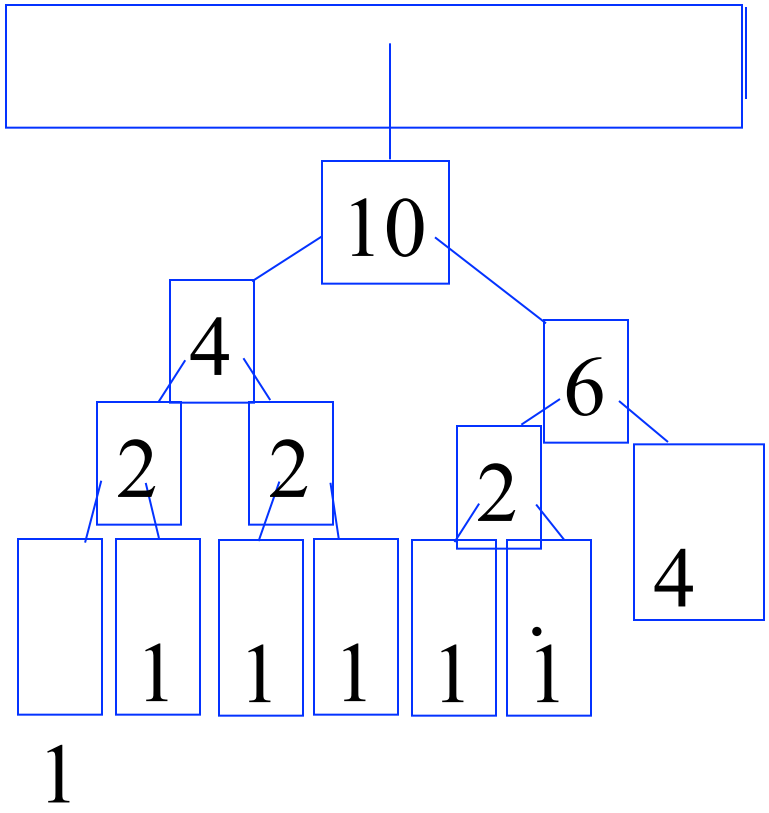
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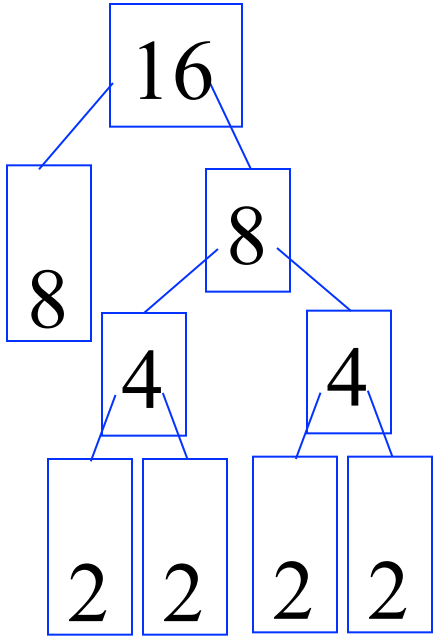
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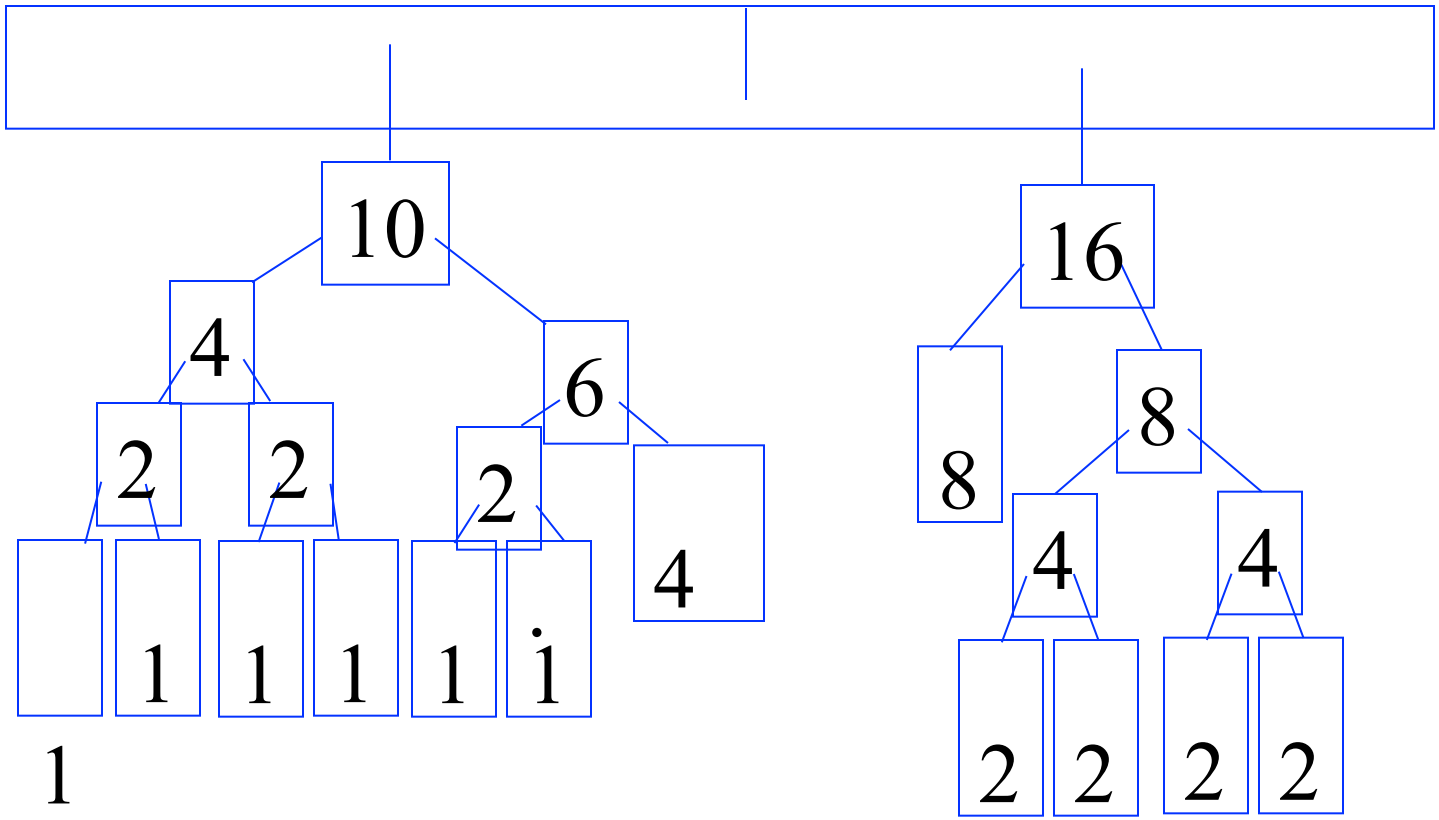
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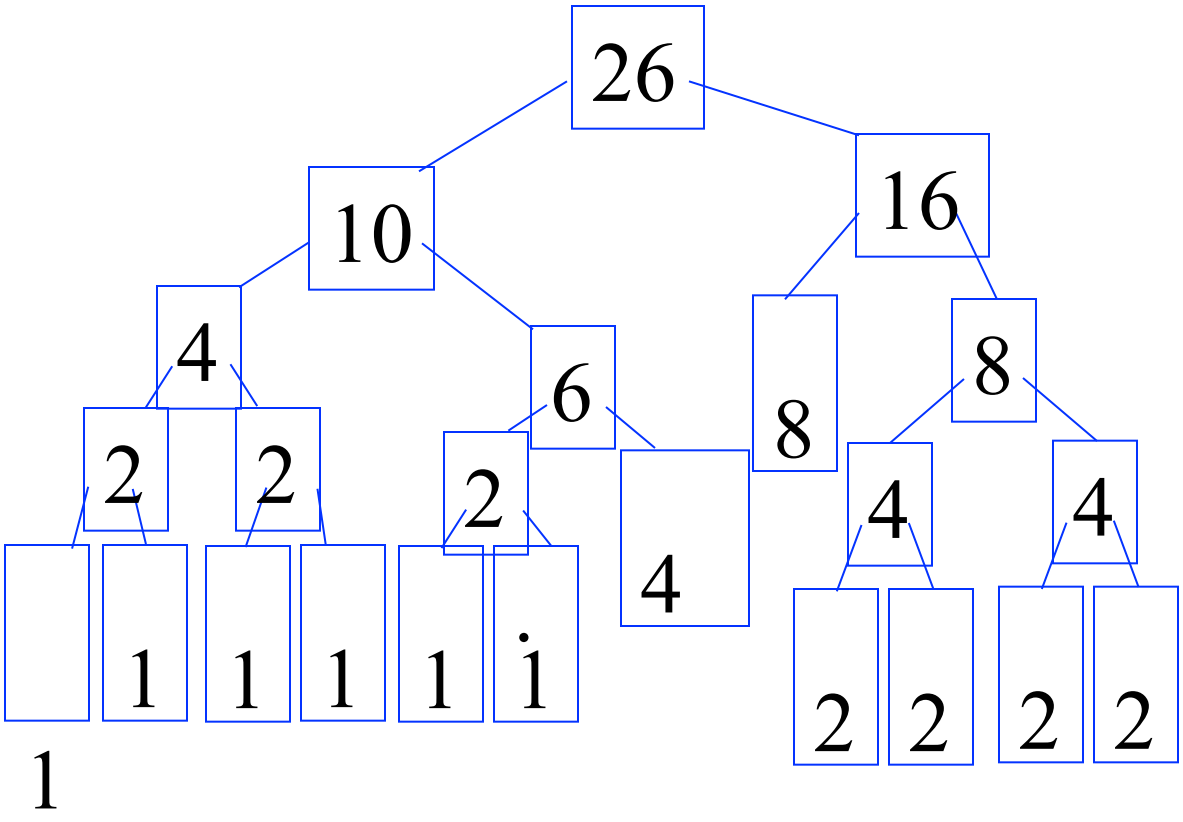
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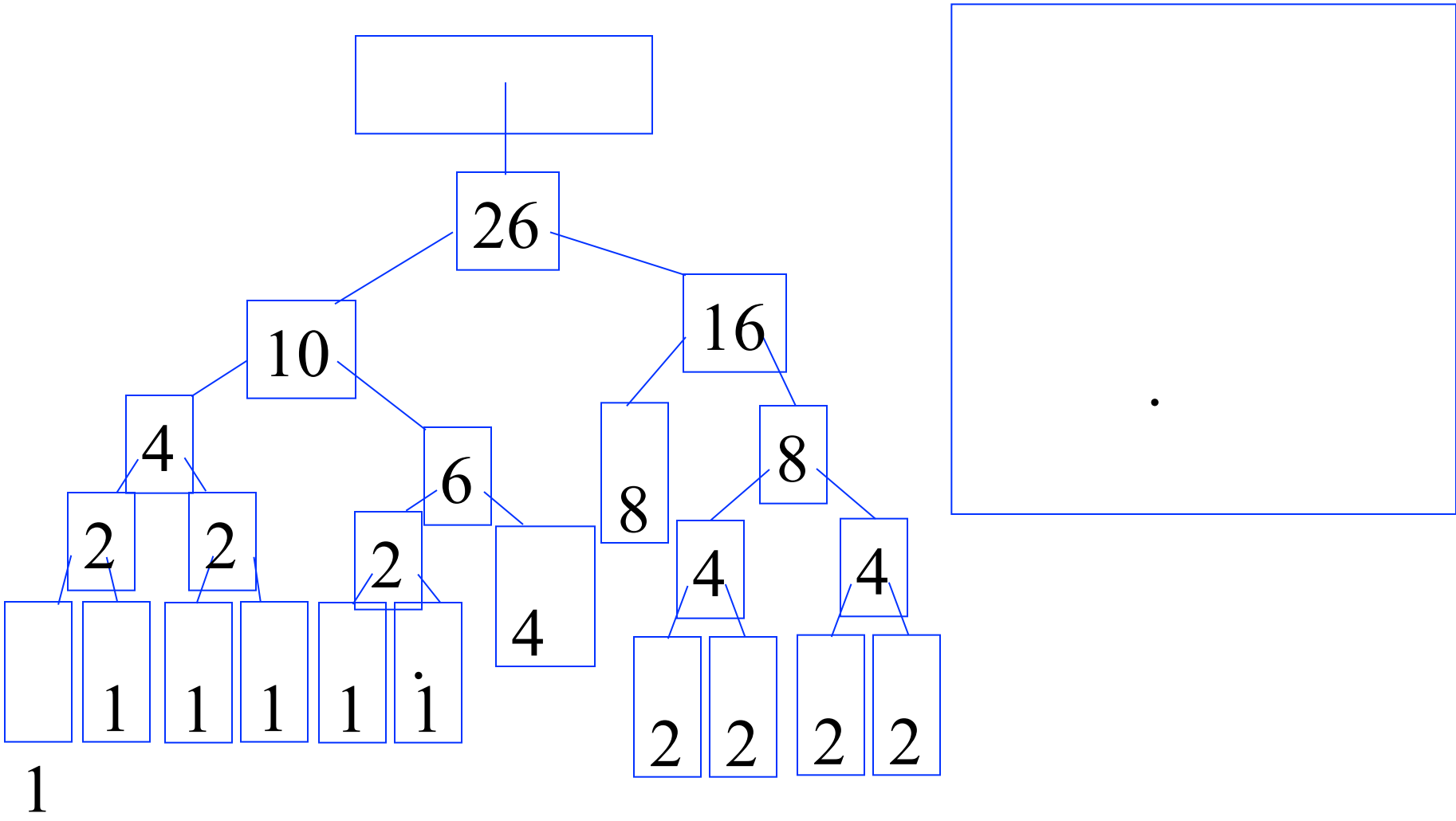
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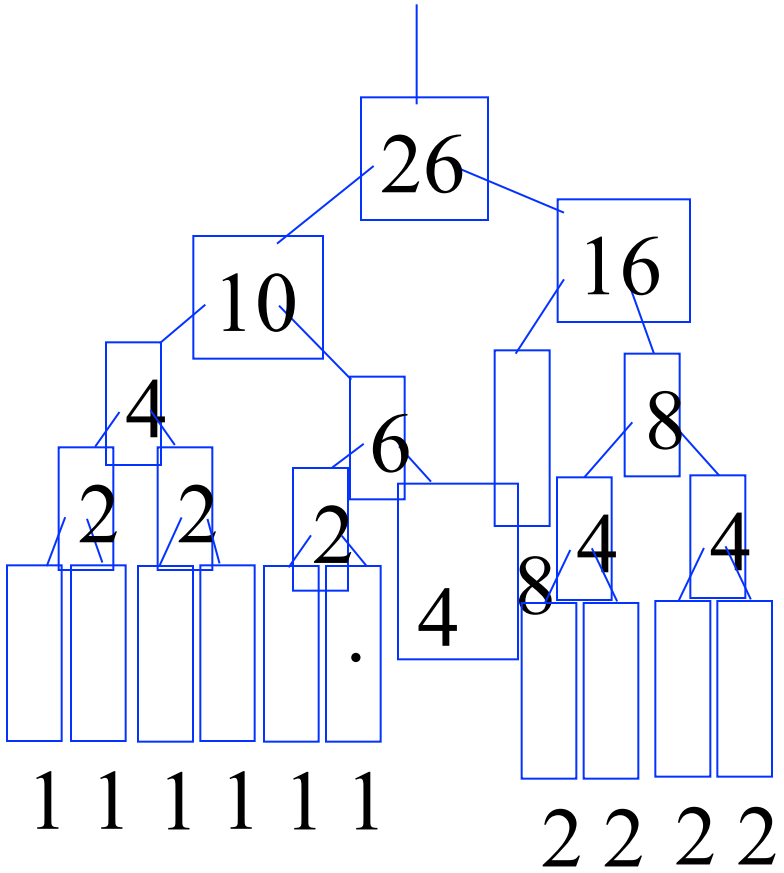

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- **Exercises: P316-3**