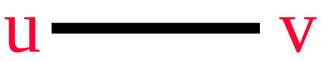
Graphs

- G = (V,E)
- V is the vertex set.
- Vertices are also called nodes and points.
- E is the edge set.
- Each edge connects two different vertices.
- Edges are also called arcs and lines.
- Directed edge has an orientation (u,v).

u → v

Graphs

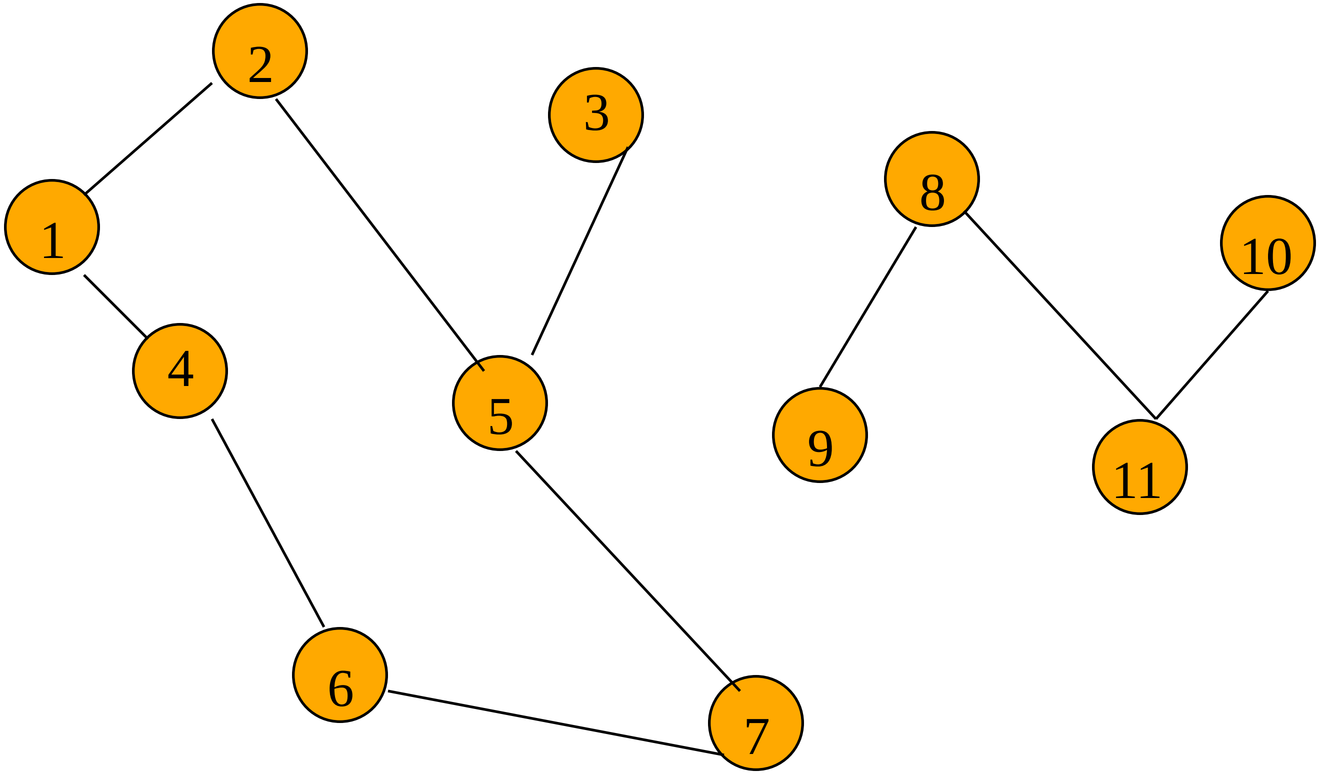
- Undirected edge has no orientation (u,v).



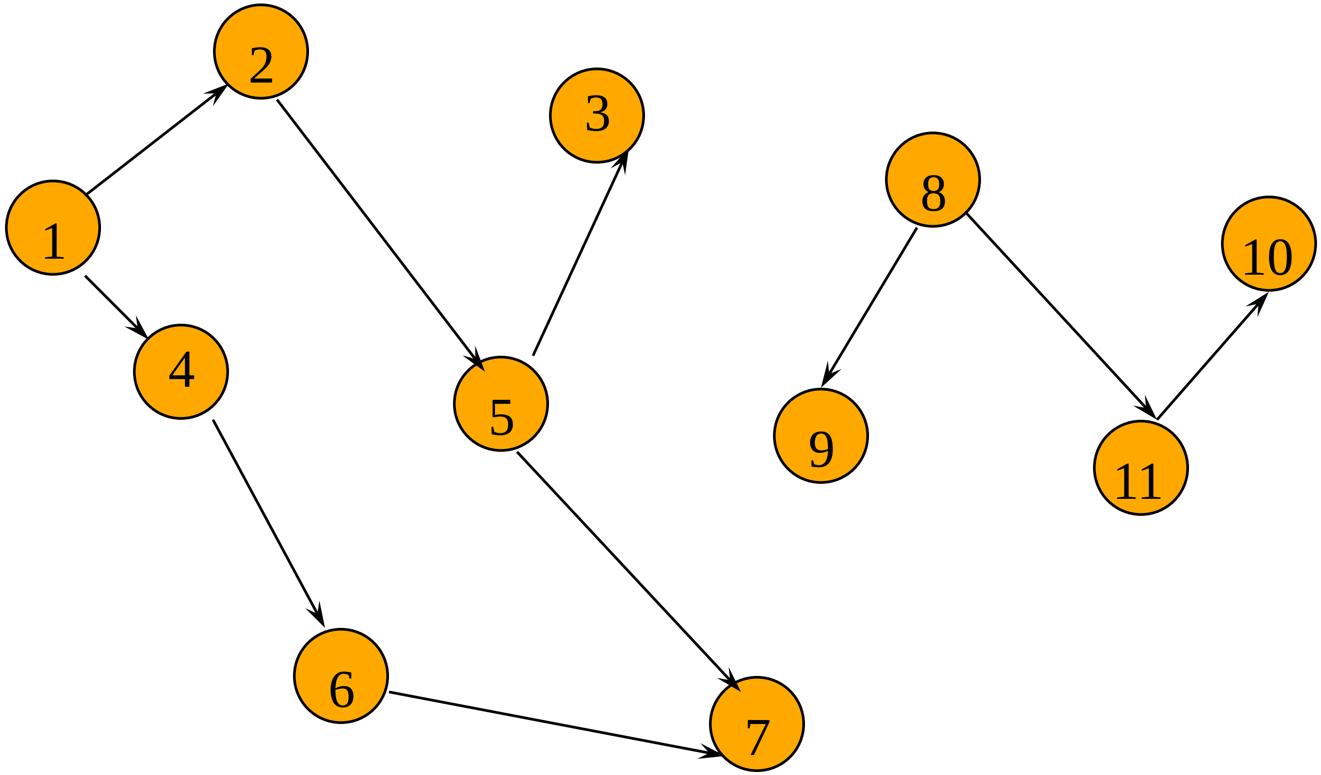
- Undirected graph => no oriented edge.
- Directed graph => every edge has an orientation.

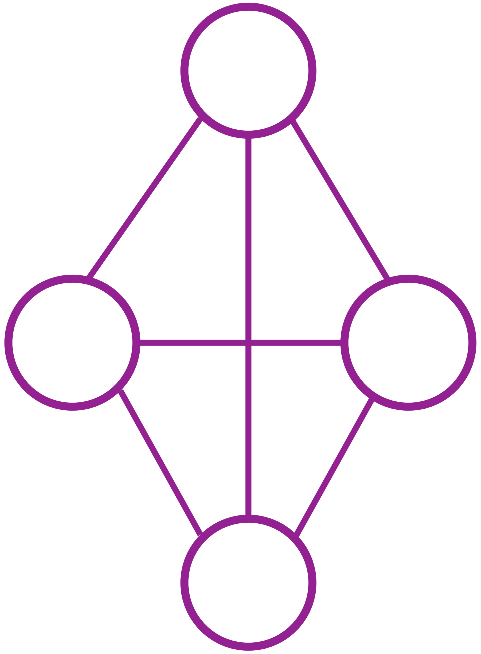
- If $(u, v) \in E(G)$, we say u and v are **adjacent** and edge (u, v) is **incident on** vertices u and v. If <u, v> is a directed edge, then vertex u is **adjacent to** v, and v is adjacent from u, <u, v> is **incident to** u and v

Undirected Graph



Directed Graph (Digraph)

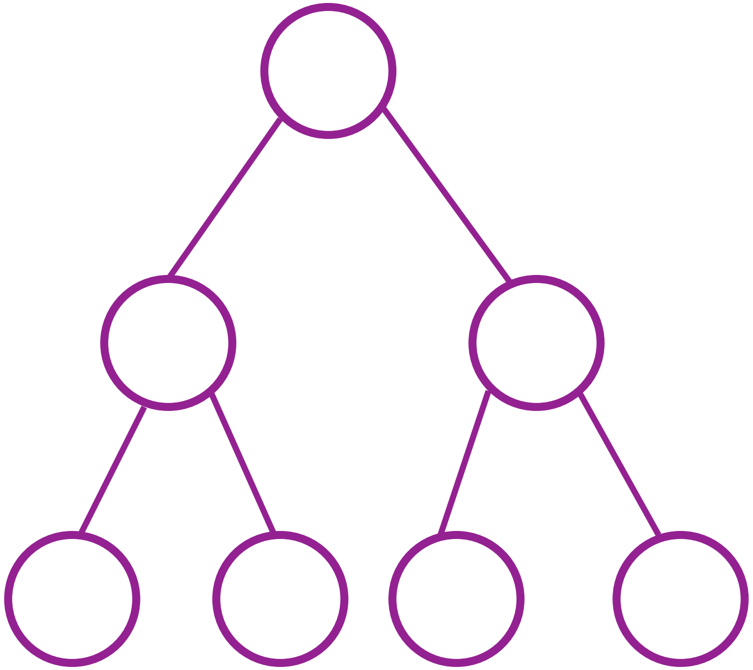




G_1:

$V(G_1)=\{0,1,2,3\}$

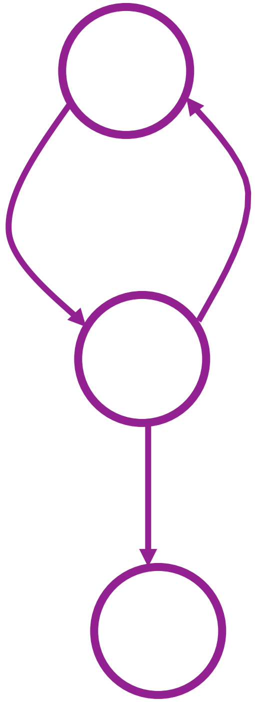
$E(G_1)=\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$



G_2:

$V(G_2)=\{0,1,2,3,4,5,6\}$

$E(G_2)=\{(0,1),(0,2),,(1,3),(1,4),(2,5),(2,6)\}$

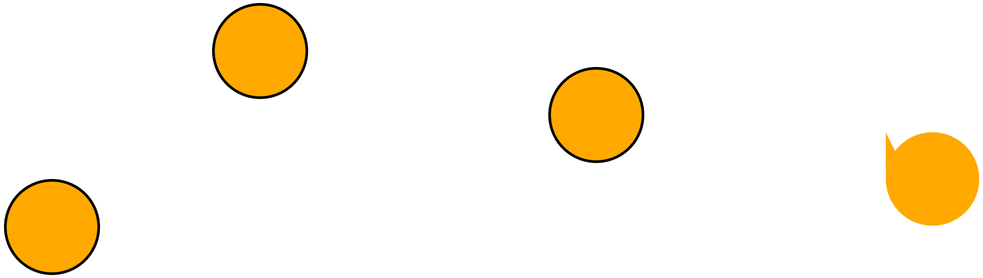


G_3:

$V(G_3) = \{0,1,2\}$

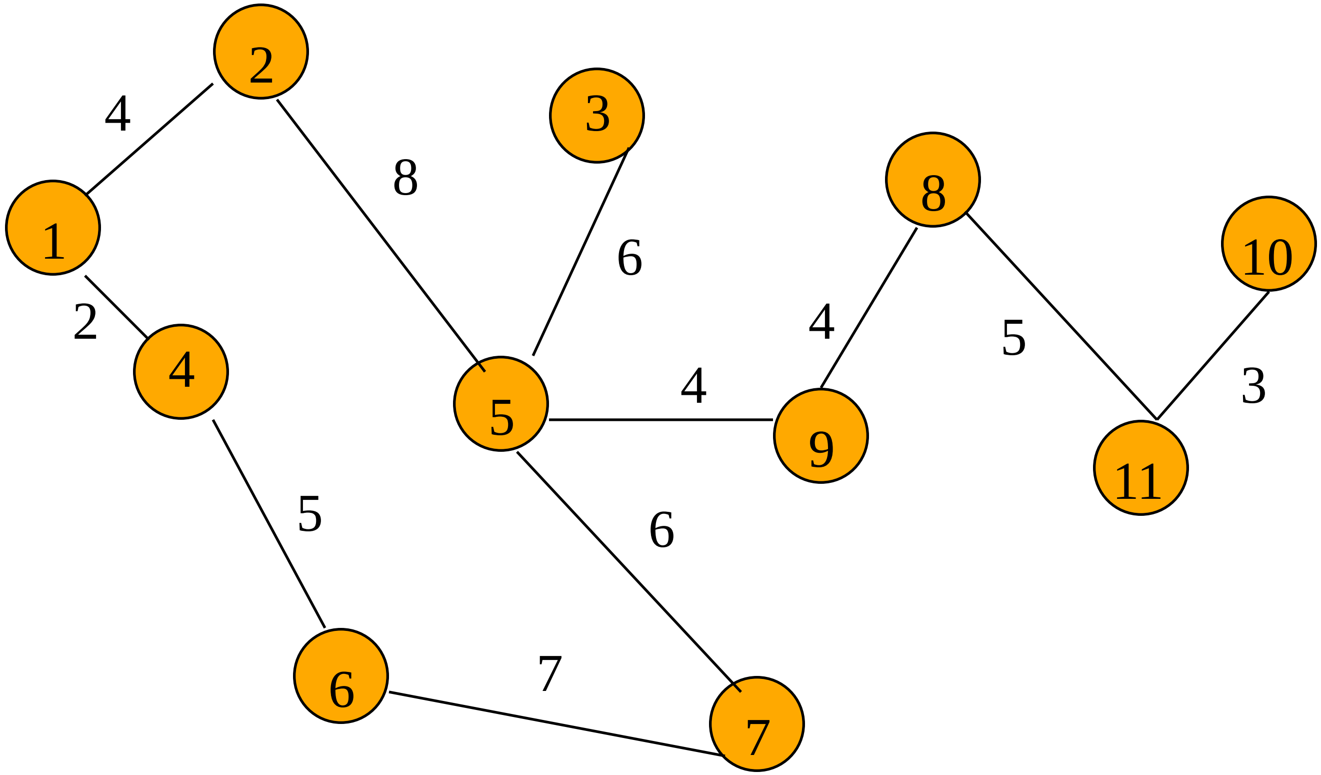
$E(G_3) = \{<0,1>,<1,0>,<1,2>\}$ (directed)

Applications—Communication Network



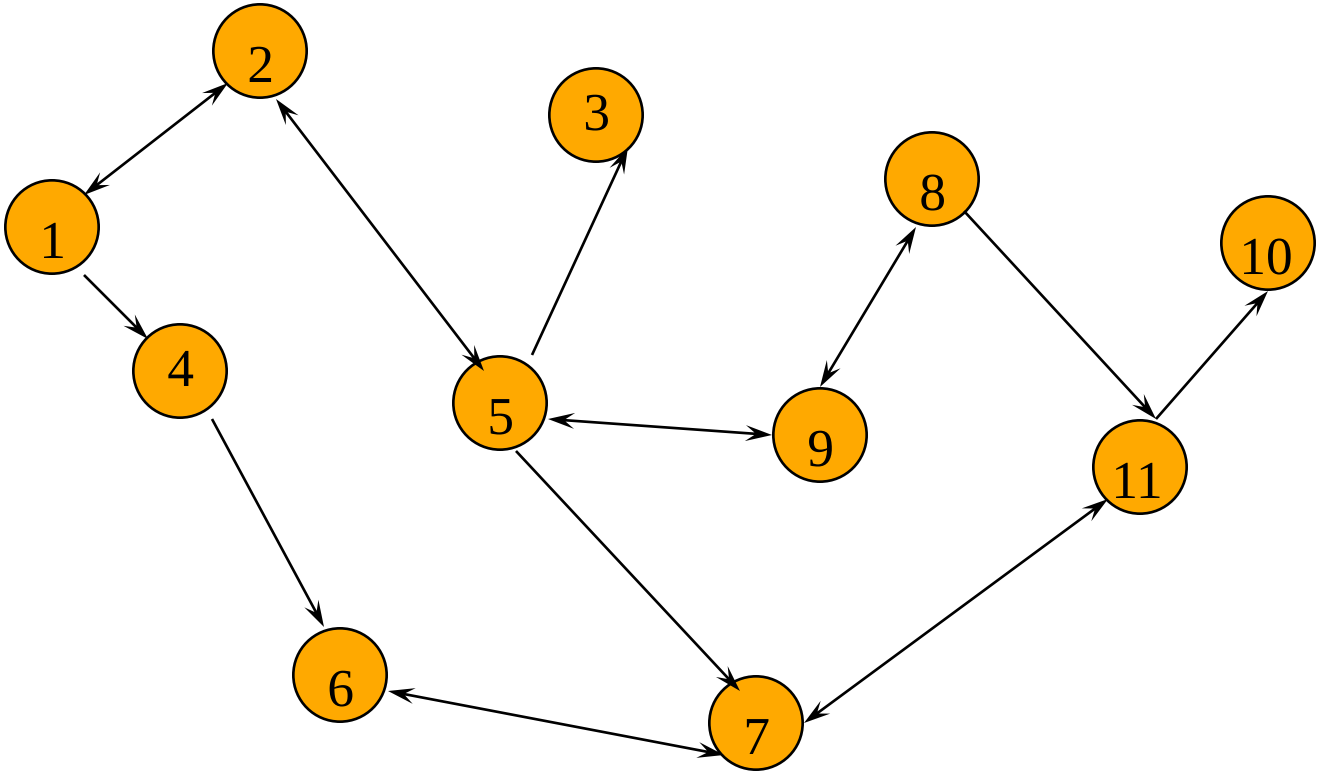
- Vertex = city, edge = communication link.

Driving Distance/Time Map



- Vertex = city, edge weight = driving distance/time.

Street Map



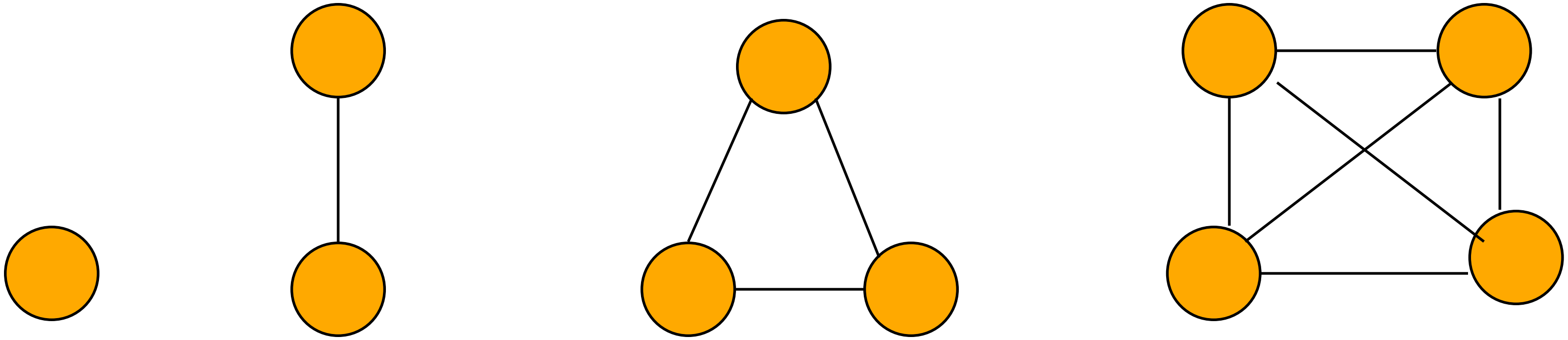
- Some streets are one way.

Restrictions:

- (v, v) or <v, v> is not legal, such edges are known as self edges
- Multiple occurrences of the same edges are not allowed. If allowed, we get a multigraph

Complete Undirected Graph

Has all possible edges.



n = 1 n = 2 n = 3 n = 4

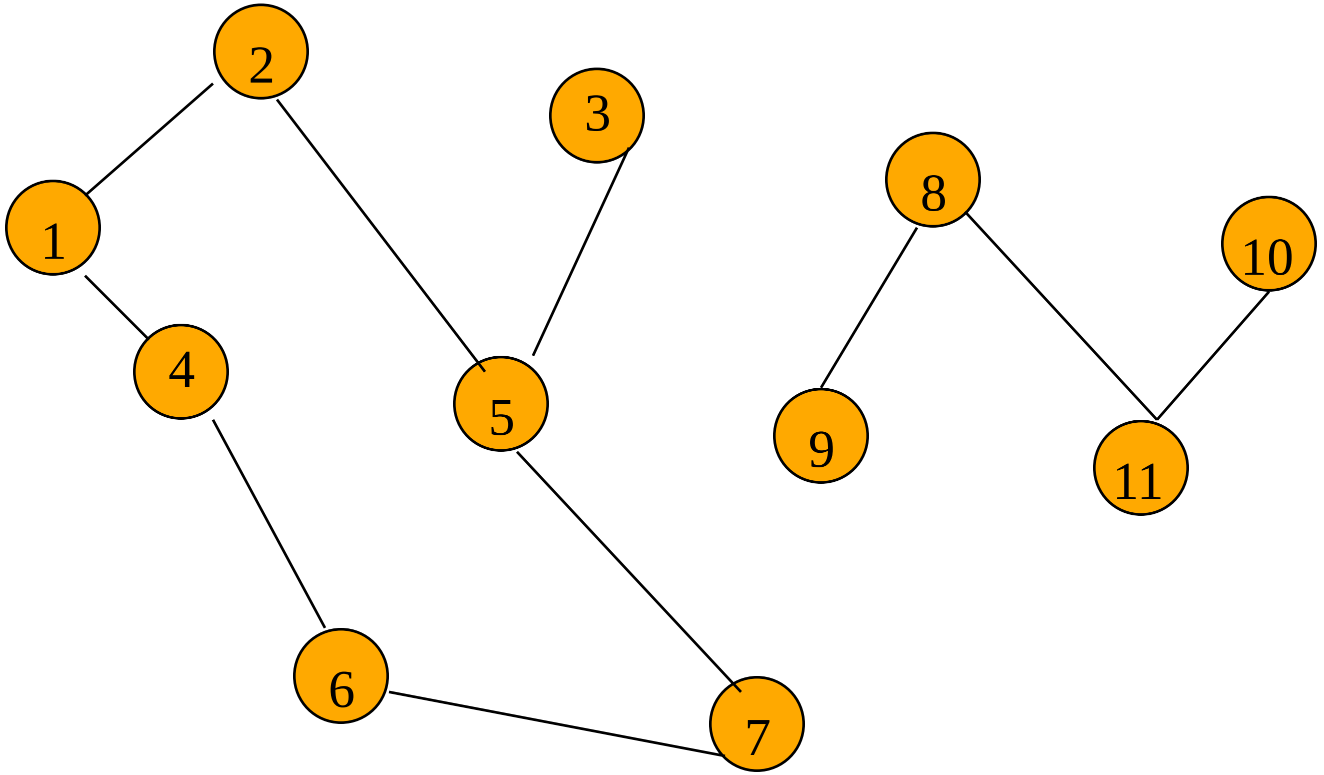
Number Of Edges—Undirected Graph

- Each edge is of the form (u,v), u != v.
- Number of such pairs in an n vertex graph is n(n-1).
- Since edge (u,v) is the same as edge (v,u), the number of edges in a complete undirected graph is n(n-1)/2.
- Number of edges in an undirected graph is <= n(n-1)/2.

Number Of Edges—Directed Graph

- Each edge is of the form (u,v), u != v.
- Number of such pairs in an n vertex graph is n(n-1).
- Since edge (u,v) is not the same as edge (v,u), the number of edges in a complete directed graph is n(n-1).
- Number of edges in a directed graph is <= n(n-1).

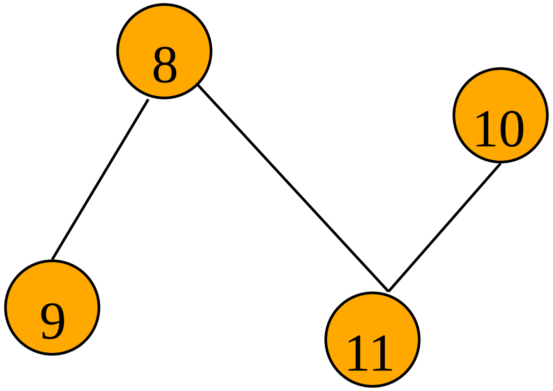
Vertex Degree



Number of edges incident to vertex.

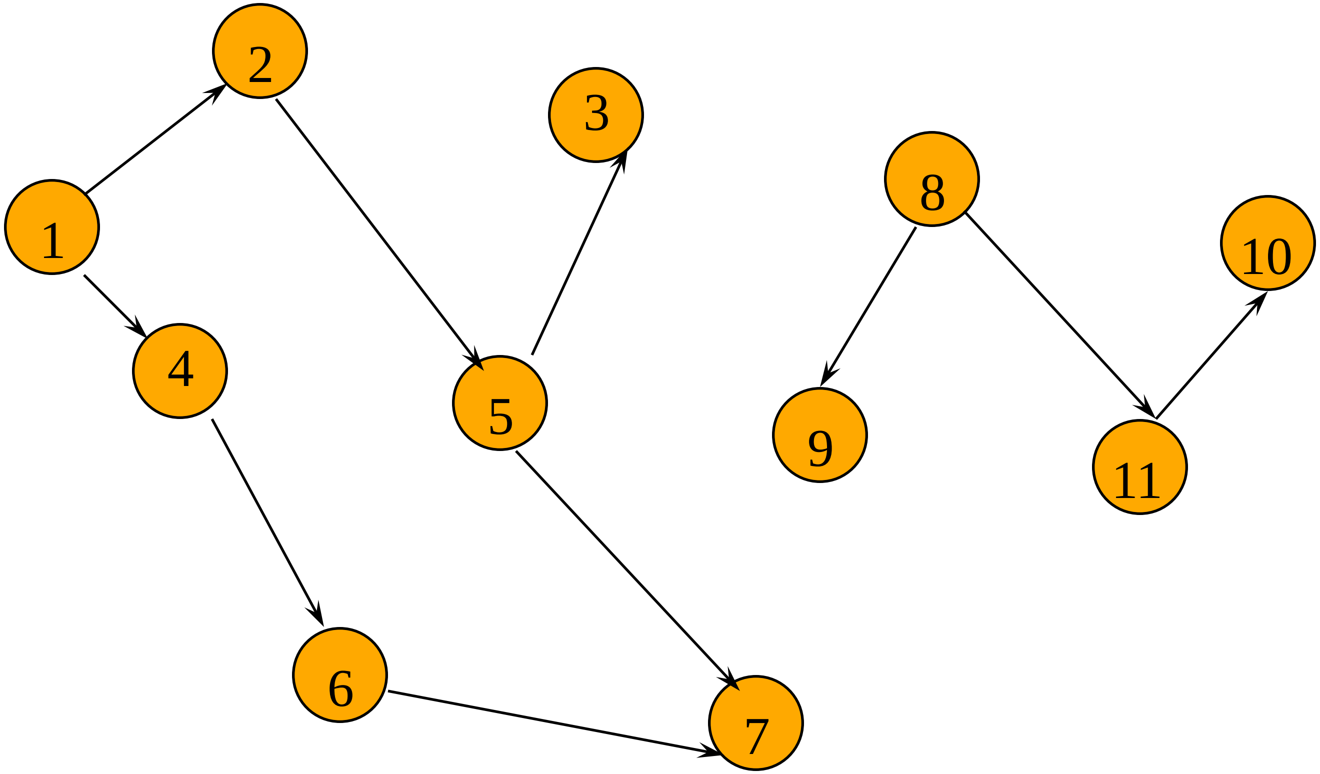
degree(2) = 2, degree(5) = 3, degree(3) = 1

Sum Of Vertex Degrees



Sum of degrees = 2e (e is number of edges)

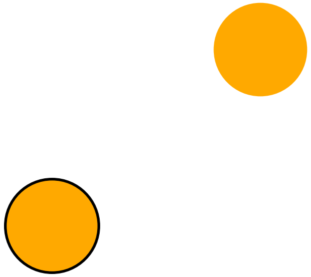
In-Degree Of A Vertex



in-degree is number of incoming edges

indegree(2) = 1, indegree(8) = 0

Out-Degree Of A Vertex



out-degree is number of outbound edges

outdegree(2) = 1, outdegree(8) = 2

Sum Of In- And Out-Degrees

each edge contributes 1 to the in-degree of some vertex and 1 to the out-degree of some other vertex

sum of in-degrees = sum of out-degrees = e,

where e is the number of edges in the digraph

Graph Operations And Representation

Notations

A **subgraph** of G is a graph G` such that V(G`) ⊆ V(G) and E(G`) ⊆ E(G).

- A **path** from u to v in G is a sequence of vertices u, i_1, i_2,…, i_k, v such that (u, i_1), (i_1, i_2),…,(i_k, v) are edges in E(G). If G` is directed, then <u, $i_1$>, <$i_1$, $i_2$>,…,<$i_k$, v> are edges in E(G`).

Notations

- A **simple path** is a path in which all vertices except possibly the first and last are distinct.
- A **cycle** is a simple path in which the first and last vertices are the same.
- For directed graph, we have **directed paths** and **cycles**.

Notations

The **length** of a path is the number of edges on it.

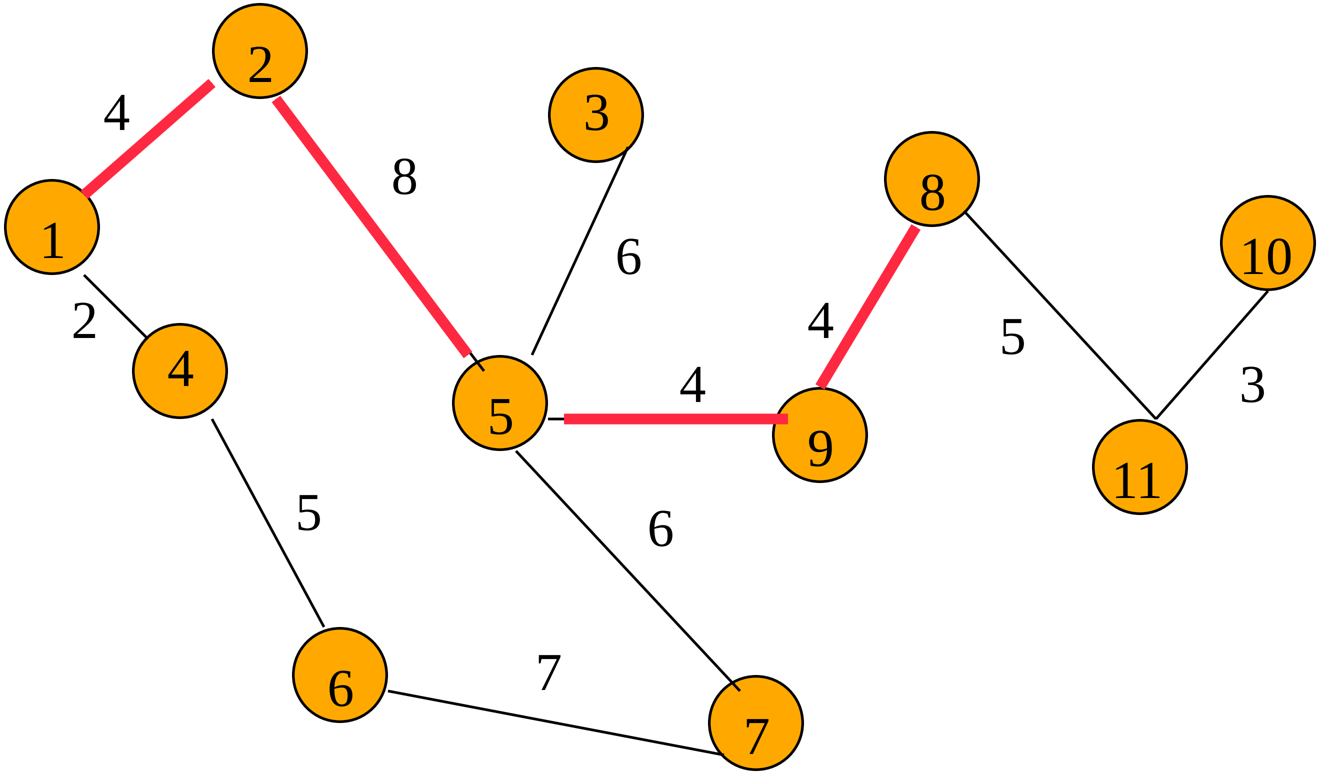
- The **length** of a path is the sum of weights of edges on it.

Sample Graph Problems

- Path problems.
- Connectedness problems.
- Spanning tree problems.

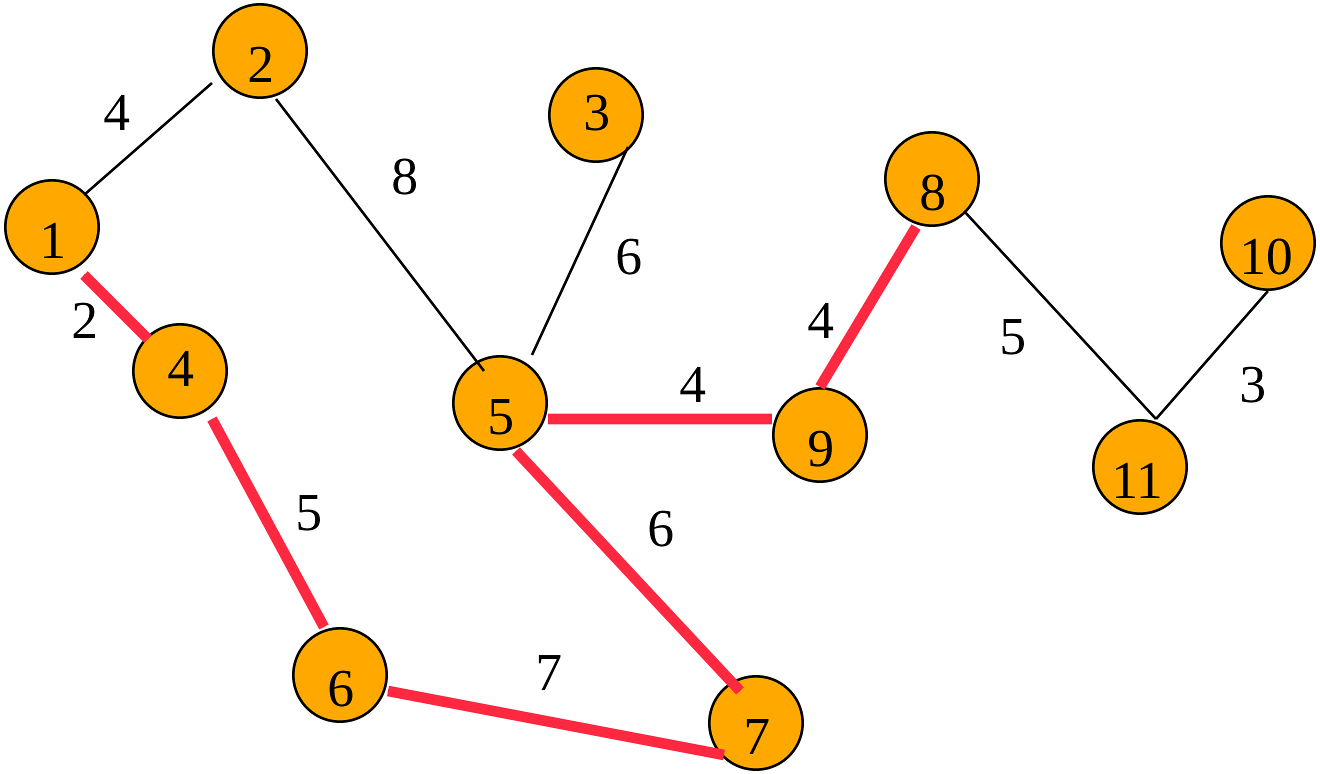
Path Finding

Path between 1 and 8.



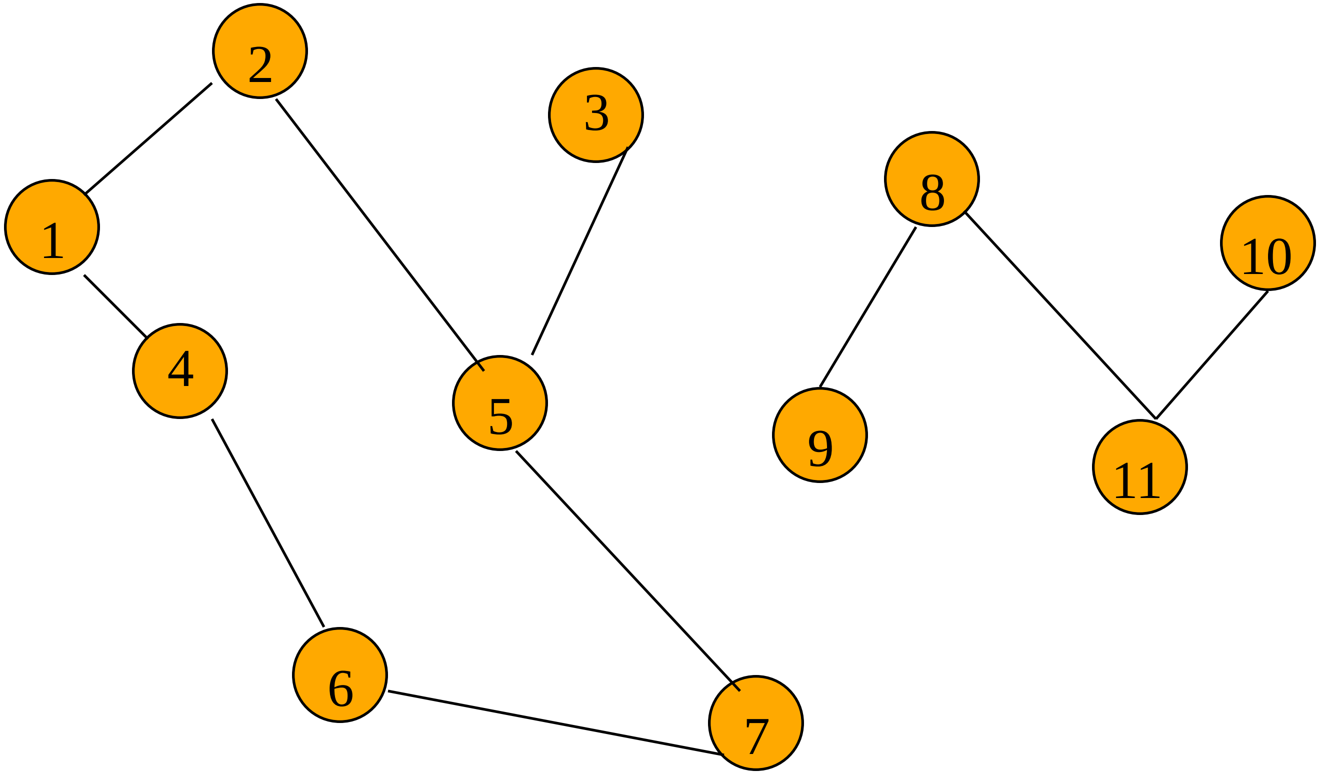
Path length is 20.

Another Path Between 1 and 8



Path length is 28.

Example Of No Path



No path between 2 and 9.

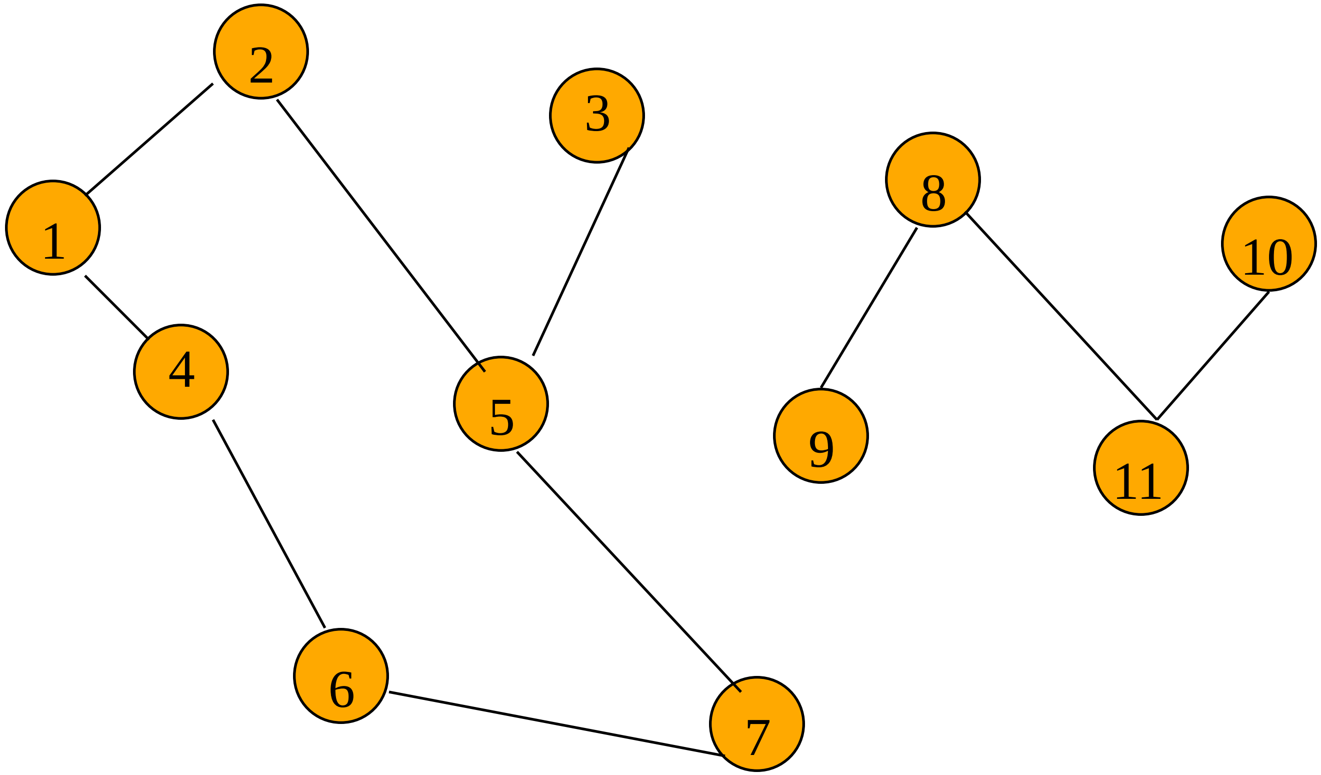
Connected Graph

- Undirected graph.
- u and v are **connected** iff there is a path in G from u to v (also from v to u)
- **Connected Graph**: There is a path between every pair of vertices.

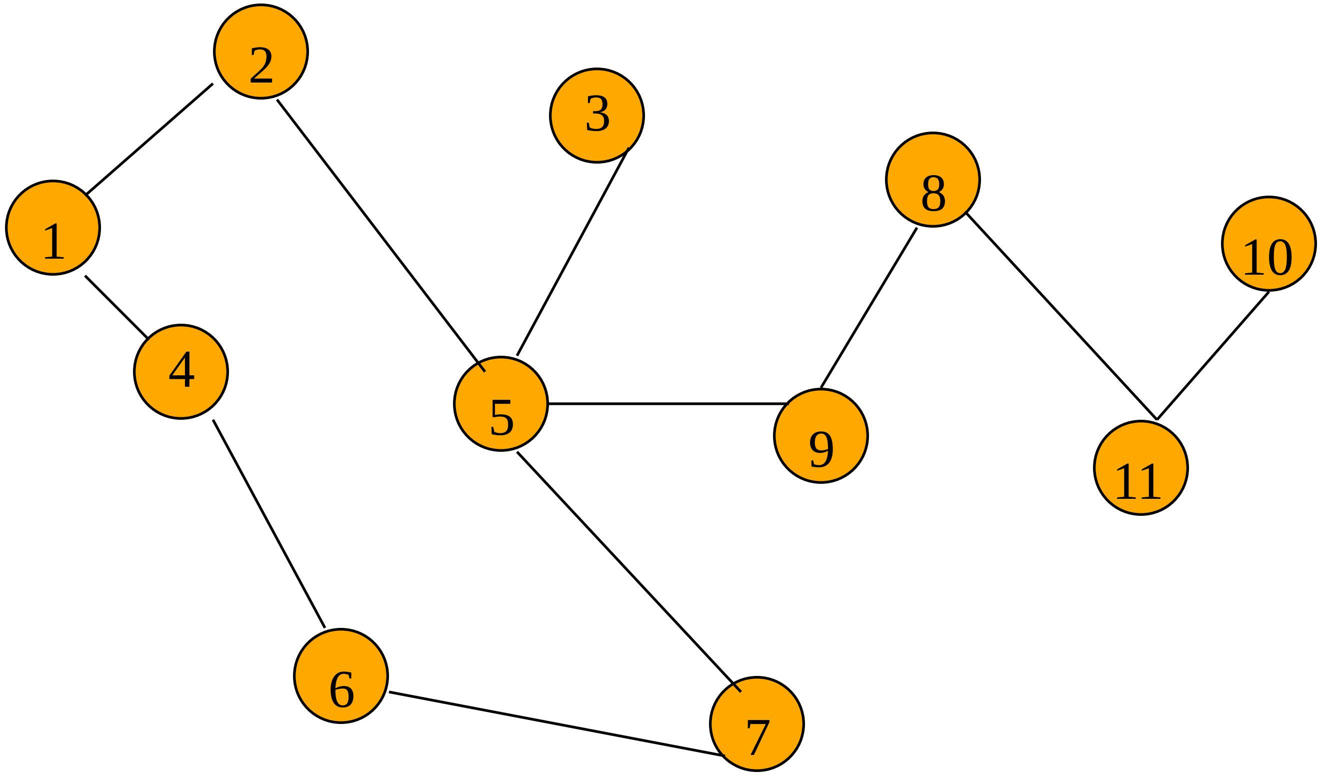
Connected Graph

- Directed graph.
- A directed G is **strongly connected** iff for every pair of distinct u and v in V(G), there is a directed path from u to v and also from v to u.
- A **strongly connected component** is a maximal subgraph that is strongly connected.

Example Of Not Connected



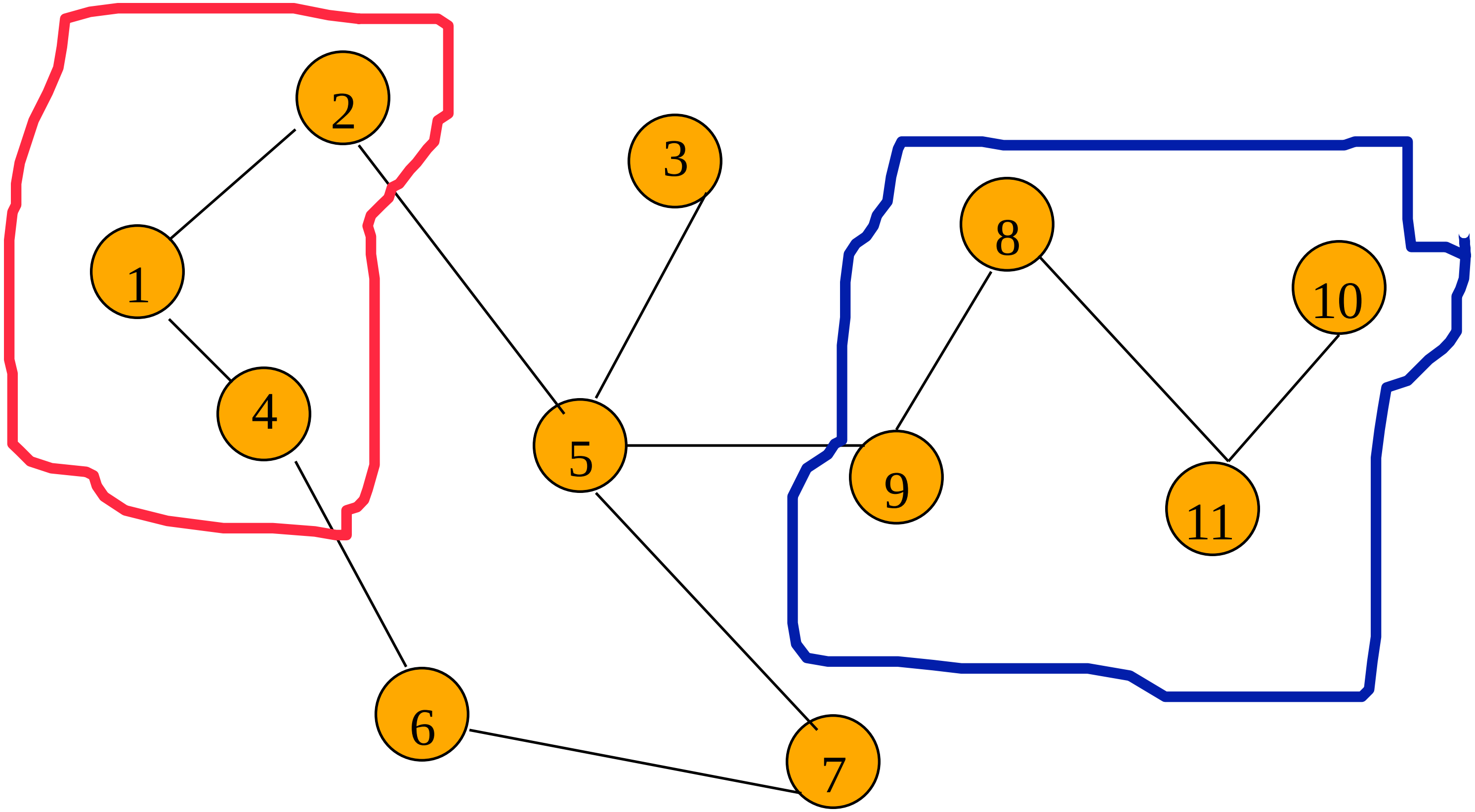
Connected Graph Example



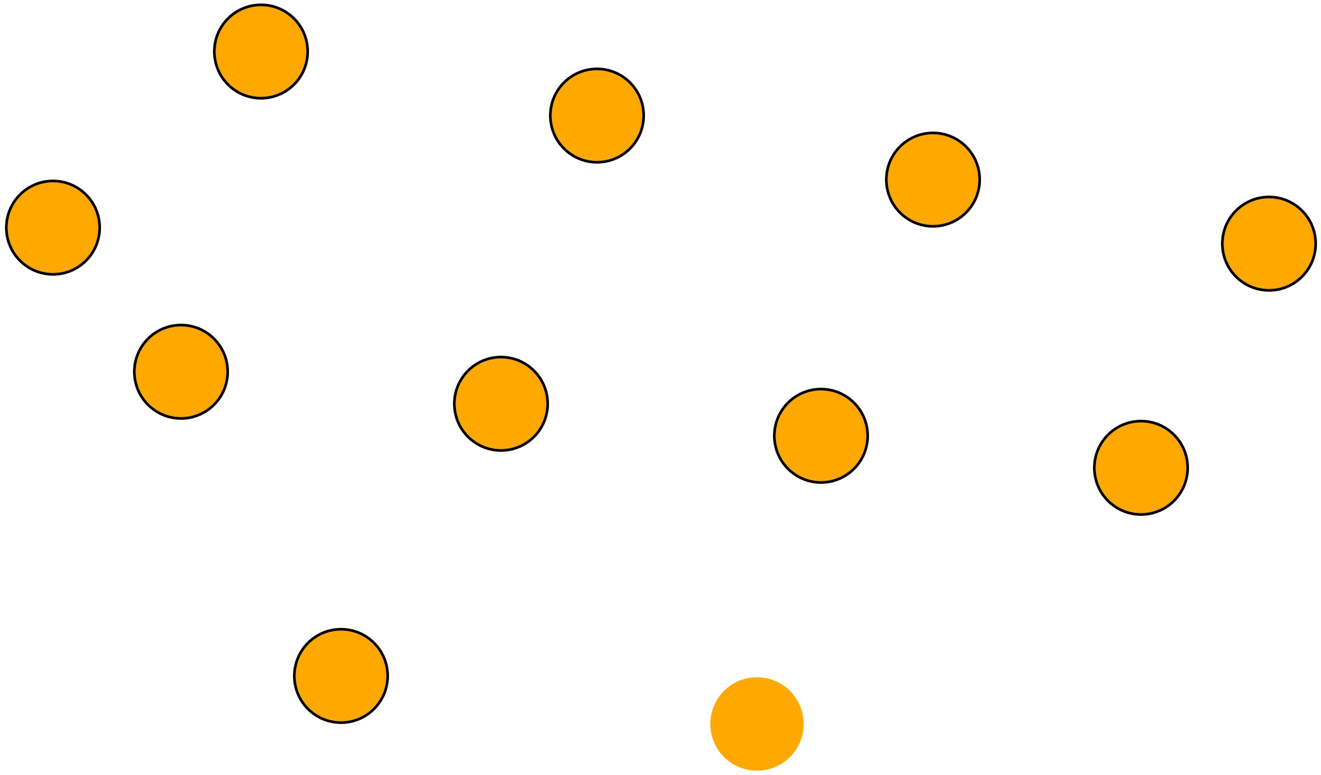
Connected Component

- A maximal subgraph that is connected.
  - Cannot add vertices and edges from original graph and retain connectedness.
- A connected graph has exactly 1 component.

Not A Component



Communication Network

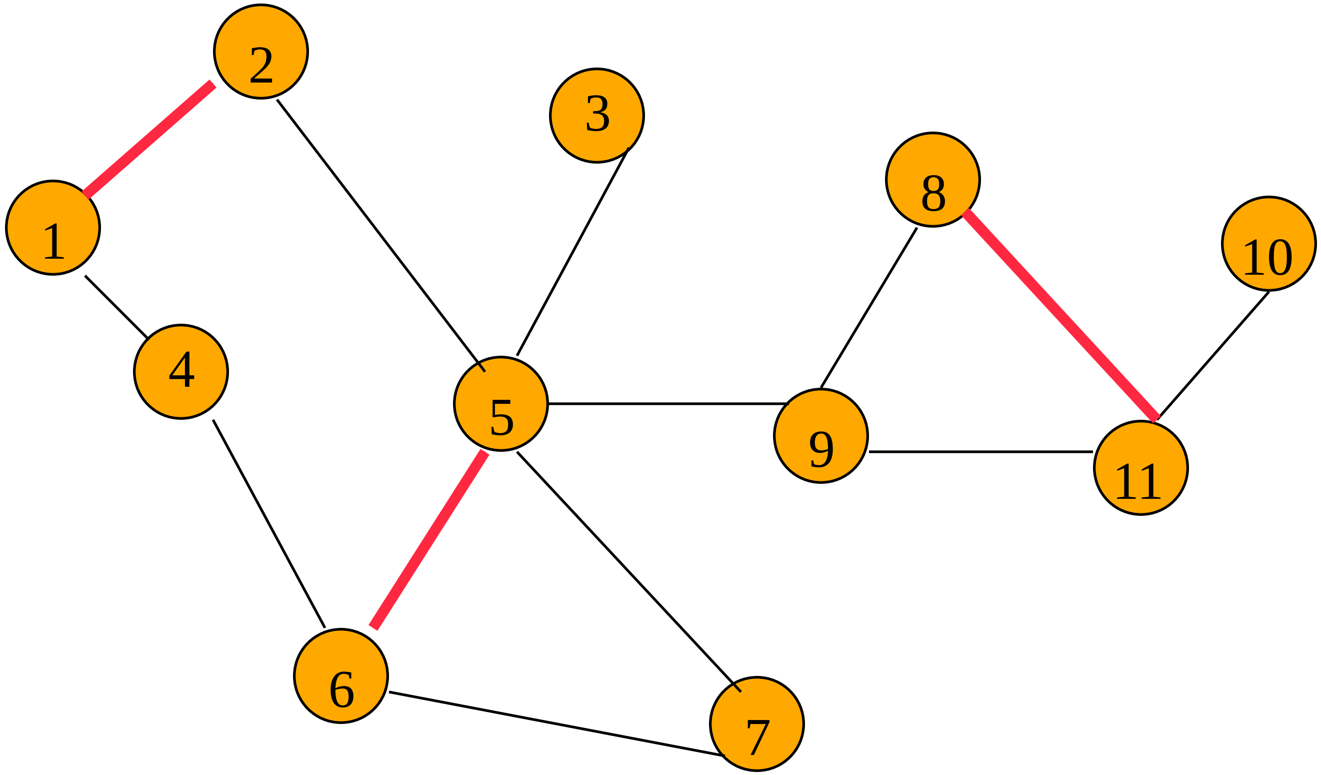


Each edge is a link that can be constructed (i.e., a feasible link).

Communication Network Problems

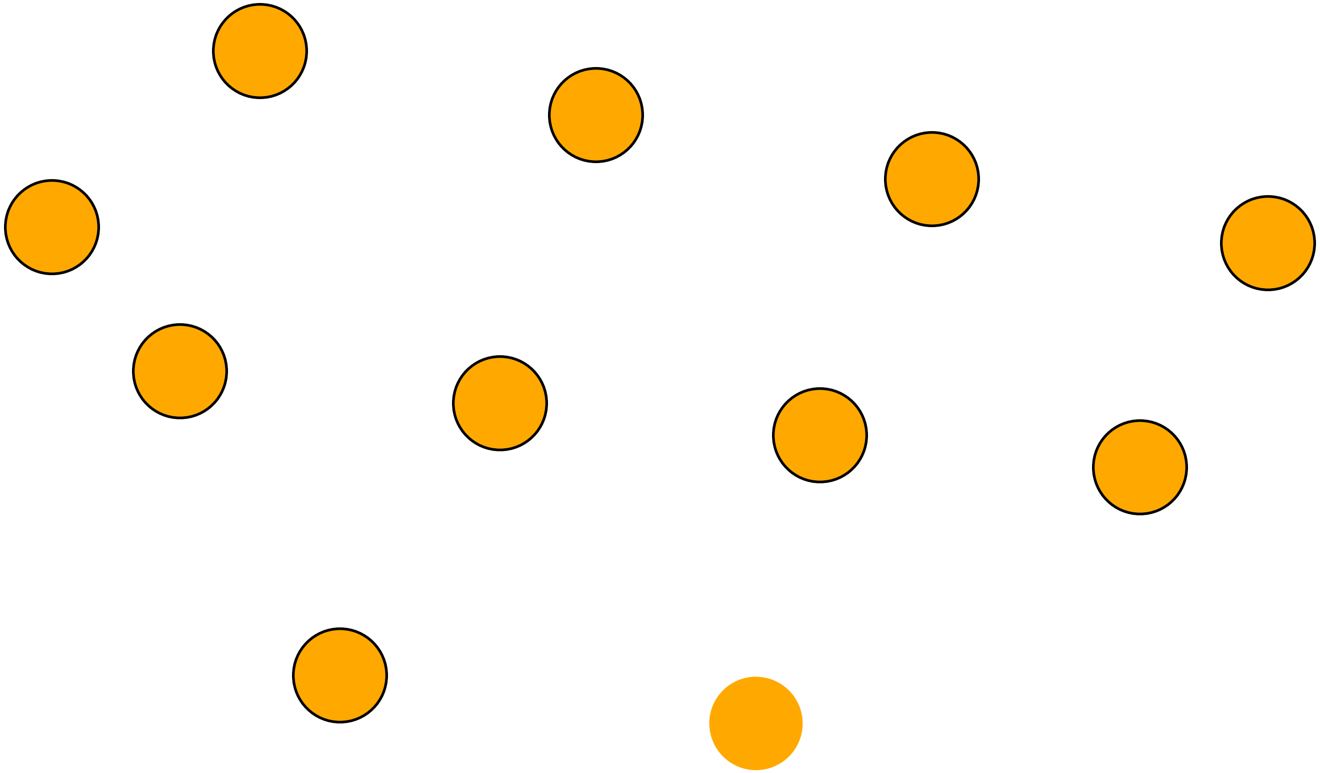
- Is the network connected?
  - Can we communicate between every pair of cities?
- Find the components.
- Want to construct smallest number of feasible links so that resulting network is connected.

Cycles And Connectedness



Removal of an edge that is on a cycle does not affect connectedness.

Cycles And Connectedness



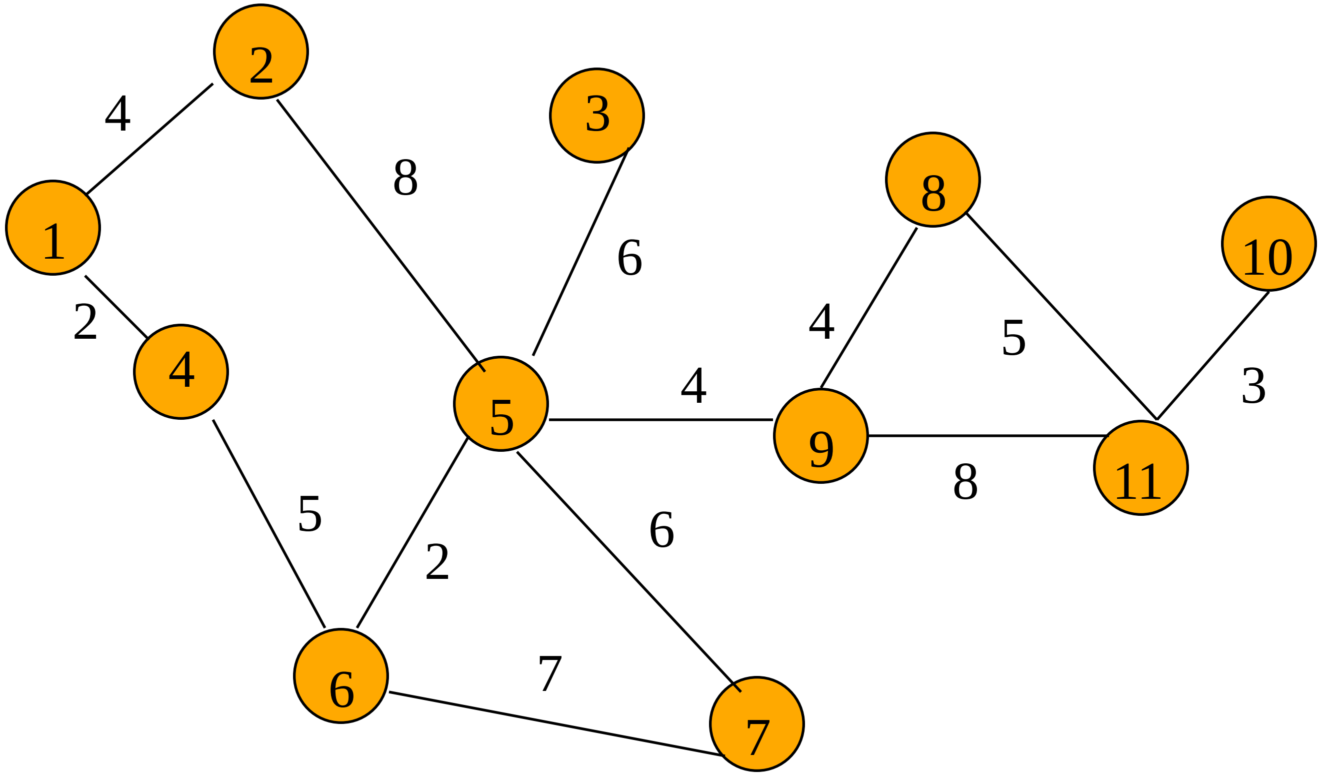
Tree

- Connected graph that has no cycles.
- n vertex connected graph with n-1 edges.

Spanning Tree

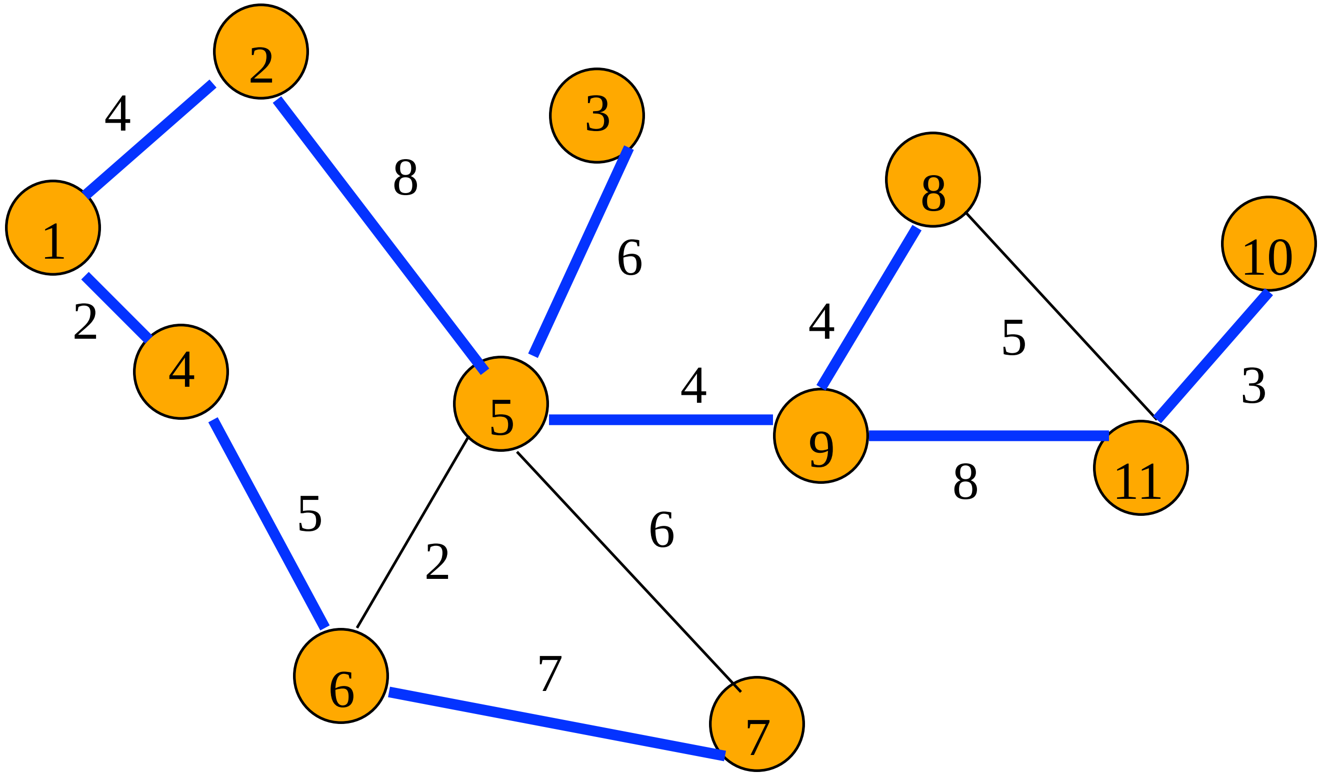
- Subgraph that includes all vertices of the original graph.
- Subgraph is a tree.
  - If original graph has n vertices, the spanning tree has n vertices and n-1 edges.

Minimum Cost Spanning Tree



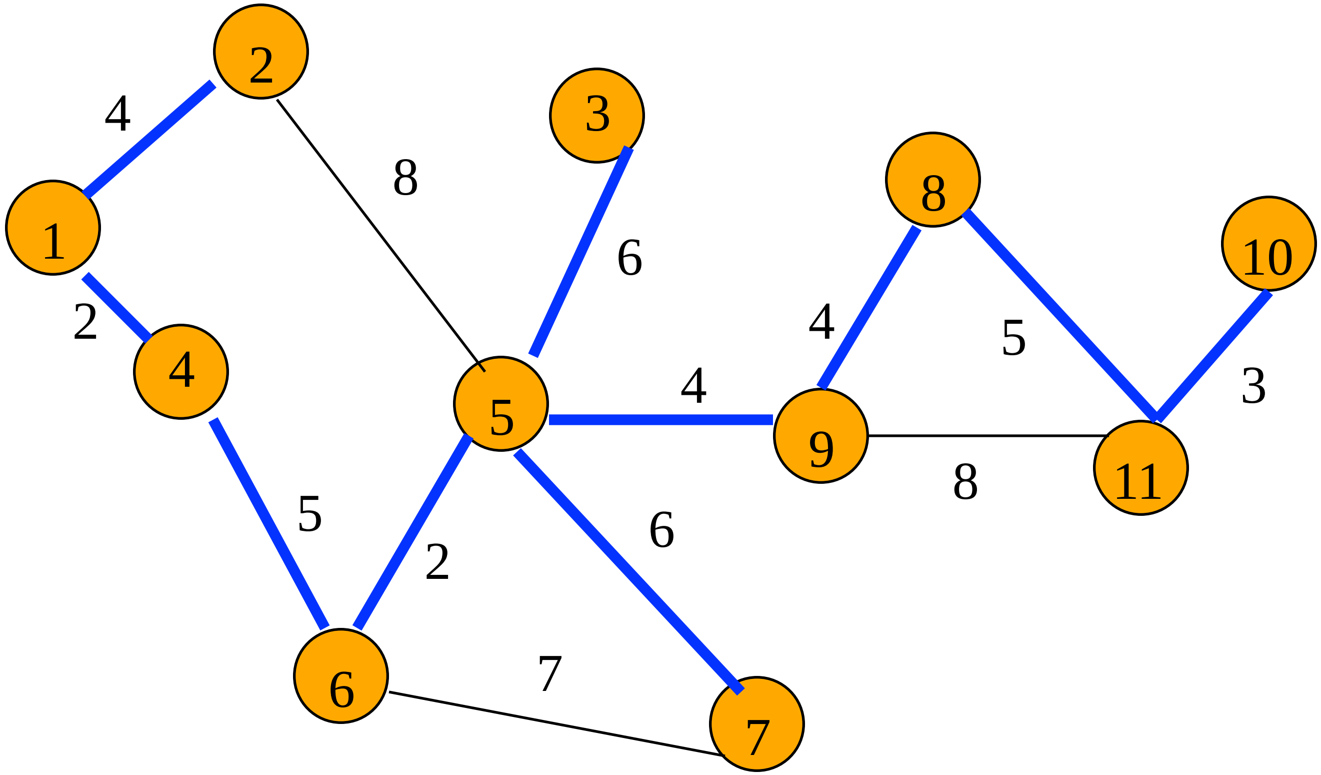
- Tree cost is sum of edge weights/costs.

A Spanning Tree



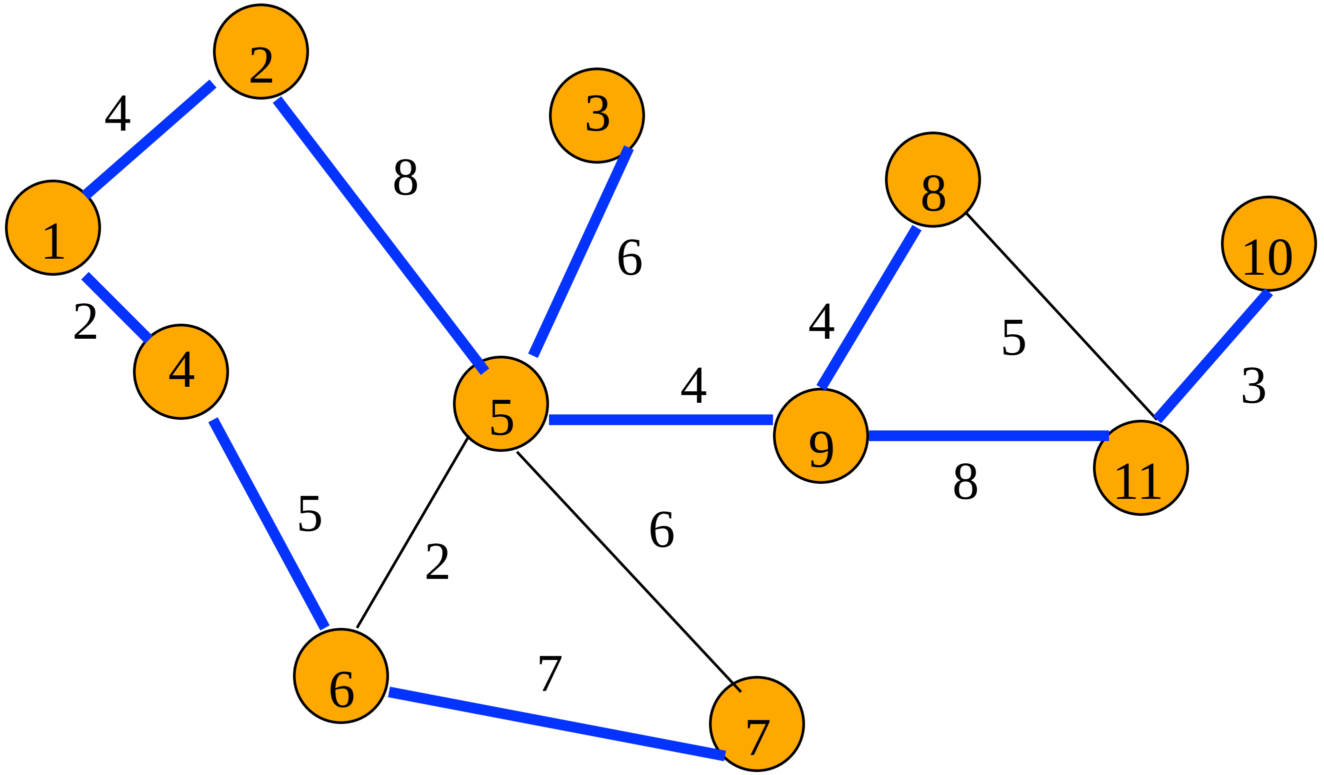
Spanning tree cost = 51.

Minimum Cost Spanning Tree



Spanning tree cost = 41.

A Wireless Broadcast Tree



Source = 1, weights = needed power.

Cost = 4 + 8 + 5 + 6 + 7 + 8 + 3 = 41.

ADT 6.1 Graph

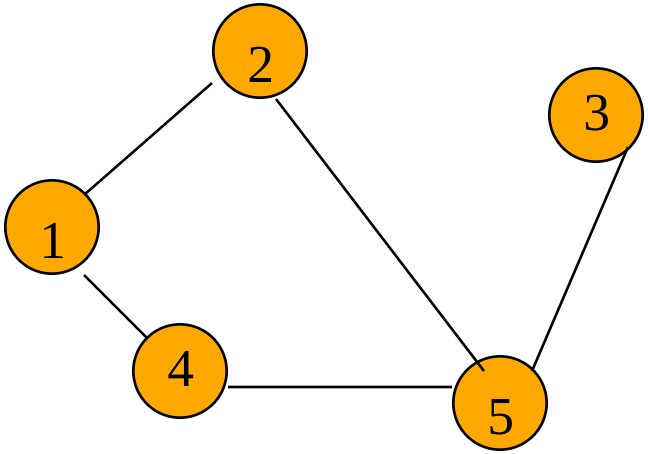
```
class Graph
{ // A non empty set of vertices and a set of undirected
  // edges, where each edge is a pair of vertices.
public:
    virtual ~Graph(){ };
       // virtual destructor
    bool IsEmpty() const {return n==0;};
       // return true iff graph has no vertices
    int NumberOfVertices() const {return n;};
       // return the number of vertices in the graph
    int NumberofEdges() const {return e;};
       // return number of edges in the graph
    virtual int Degree(int u) const =0;
       // return number of edges incident to vertex u
    virtual bool ExisteEdge(int u, int v) const =0;
       // return true iff graph has edge (u, v)
    virtual void InsertVertex (int v) =0;
       // insert vertex v into graph, v has no incident edges
    virtual void InsertEdge (int u, int v) =0;
       // insert edge (u, v) into graph
    virtual void DeleteVertex (int v);
       // delete v and all edges incident to it
    virtual void DeleteEdge (int u, int v) =0;
       // delete edge (u, v) from the graph
private:
     int n;    // number of vertices
     int e;    // number of edges
};
```

Graph Representation

- Adjacency Matrix
- Adjacency Lists
  - Linked Adjacency Lists
  - Array Adjacency Lists

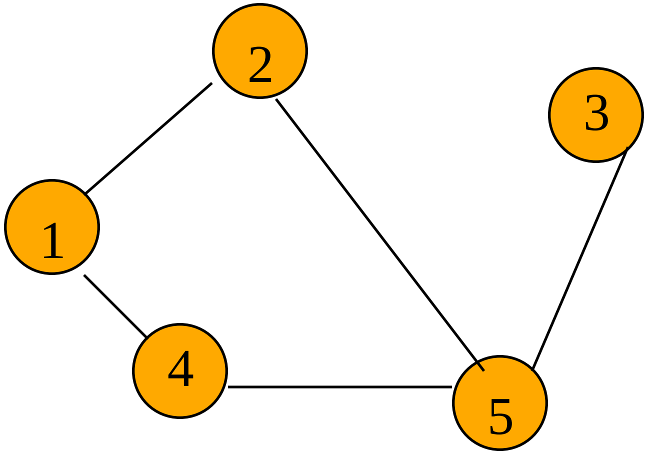
Adjacency Matrix

- 0/1 n x n matrix, where n = # of vertices
- A(i,j) = 1 iff (i,j) is an edge



	1	2	3	4	5
1	0	1	0	1	0
2	1	0	0	0	1
3	0	0	0	0	1
4	1	0	0	0	1
5	0	1	1	1	0

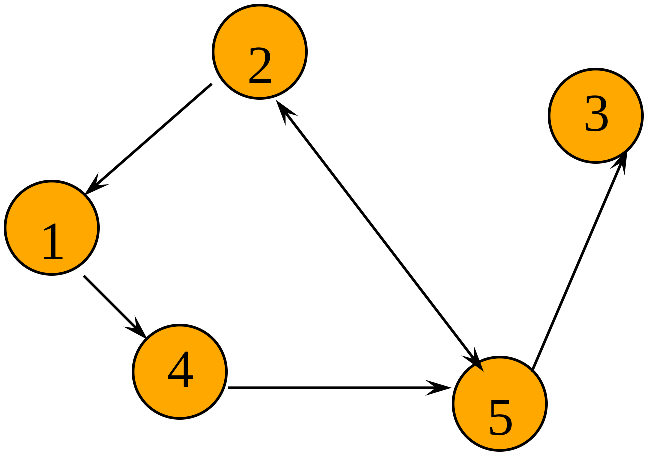
Adjacency Matrix Properties



$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 & 0 & 1 \\
4 & 1 & 0 & 0 & 0 & 1 \\
5 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

- Diagonal entries are zero.
- Adjacency matrix of an undirected graph is symmetric.
  - A(i,j) = A(j,i) for all i and j.

Adjacency Matrix (Digraph)



$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 1 & 1 & 0 & 0
\end{array}
$$

- Diagonal entries are zero.
- Adjacency matrix of a digraph need not be symmetric.

Adjacency Matrix

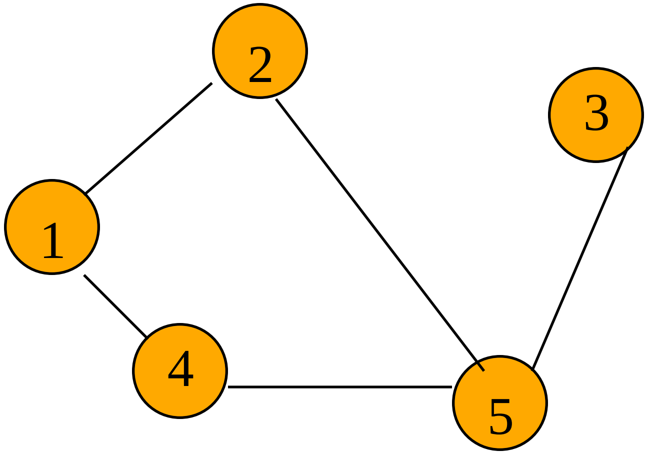
- n^2 bits of space
- For an undirected graph, may store only lower or upper triangle (exclude diagonal).
    - (n-1)n/2 bits
- time to find vertex degree and/or vertices adjacent to a given vertex？
    - O(n)

Adjacency Matrix

- For an graph
- d(i) = $\sum_{j=0}^{n-1} a[i][j]$
- For a digraph
- out-d(i) = $\sum_{j=}^{n-} a \; i \quad j$
- in-d(j) = $\sum_{i=}^{-} a \; i \quad j$

Adjacency Lists

- Adjacency list for vertex i is a linear list of vertices adjacent from vertex i.
- An array of n adjacency lists.



aList[1] = (2,4)

aList[2] = (1,5)

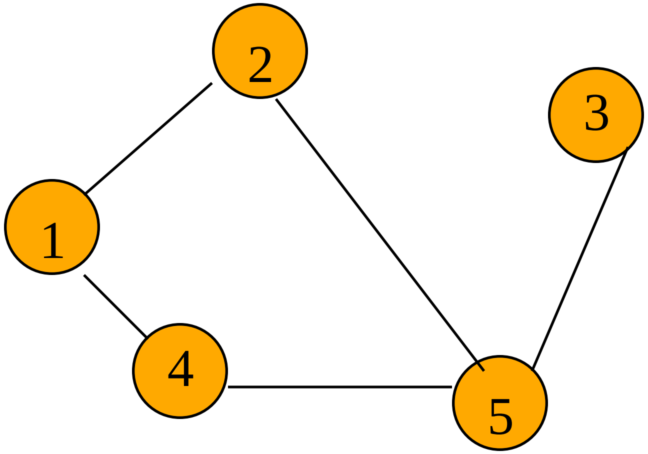
aList[3] = (5)

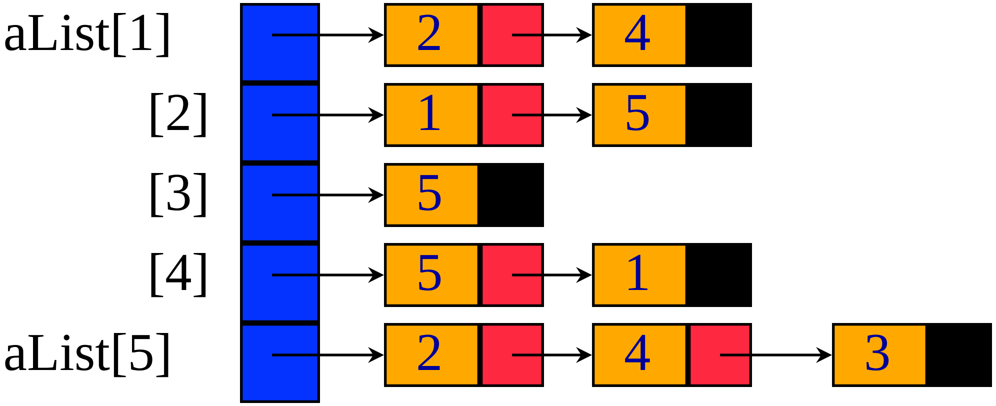
aList[4] = (5,1)

aList[5] = (2,4,3)

Linked Adjacency Lists

- Each adjacency list is a chain.





Array Length = n

# of chain nodes = 2e (undirected graph)

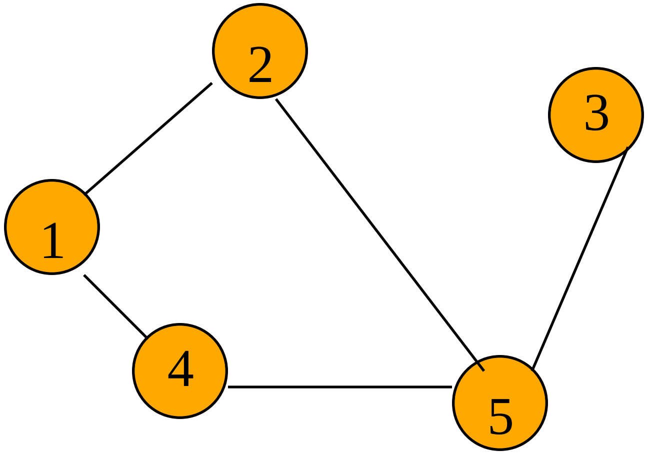
# of chain nodes = e (digraph)

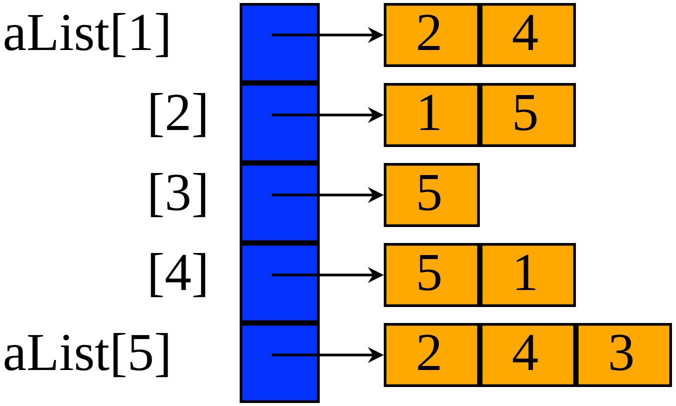
Linked Adjacency Lists

```
class LinkedGraph {
public:
    LinkedGraph (const int vertices): e(0) {
       if (vertices < 1) throw “Number of vertices must be > 0”;
       n = vertices;
       adjLists = new Chain<int>[n];
    };
private:
    Chain<int>* adjLists;
    int n;
    int e;
};
```

Array Adjacency Lists

- Each adjacency list is an array list.





aList[1] → 2 4

[2] → 1 5

[3] → 5

[4] → 5 1

aList[5] → 2 4 3

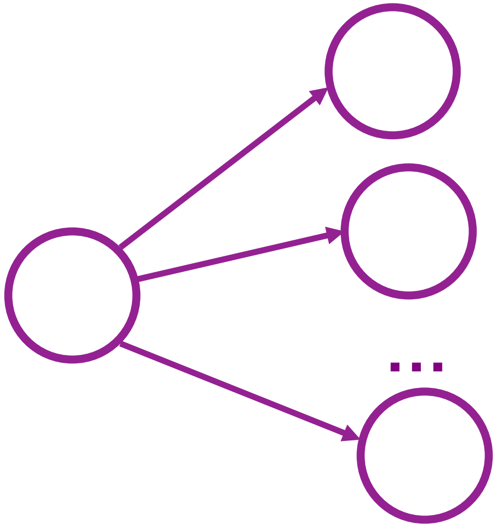
Array Length = n

# of list elements = 2e (undirected graph)

# of list elements = e (digraph)

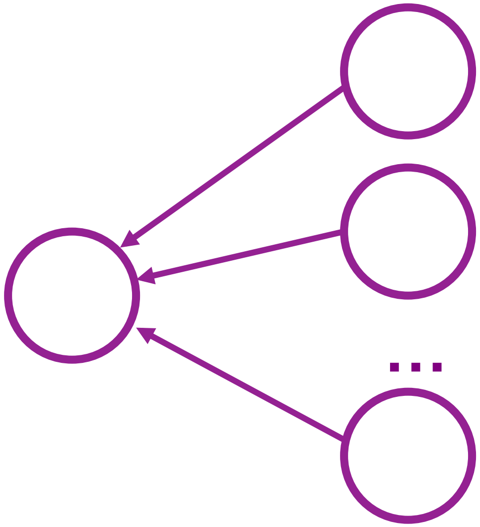
Adjacency Lists

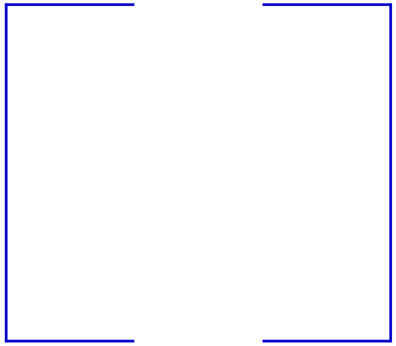
- Digraph

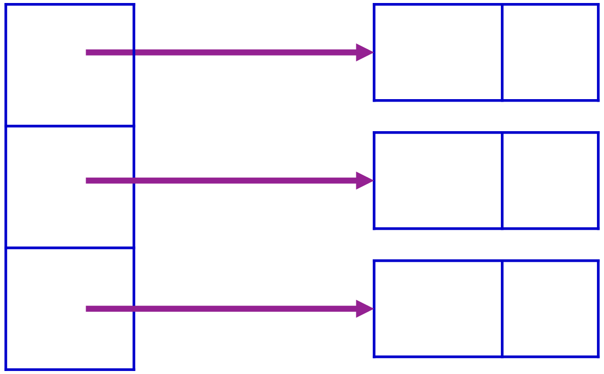


Inverse Adjacency Lists

- Digraph



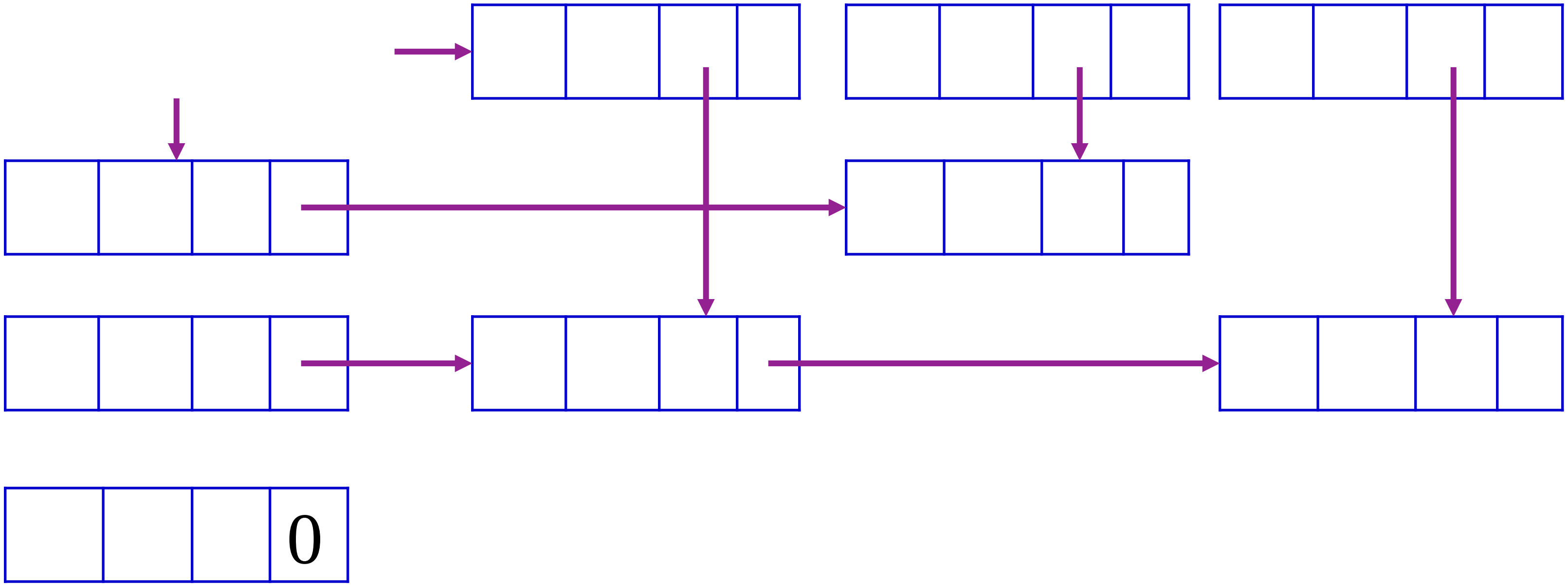




Adjacency Lists

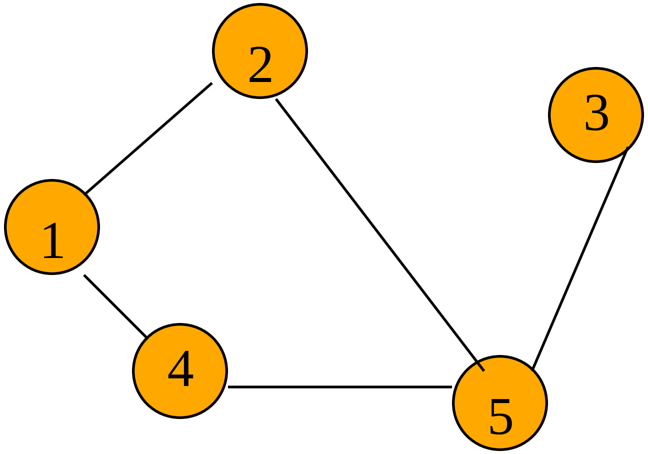
- Digraph

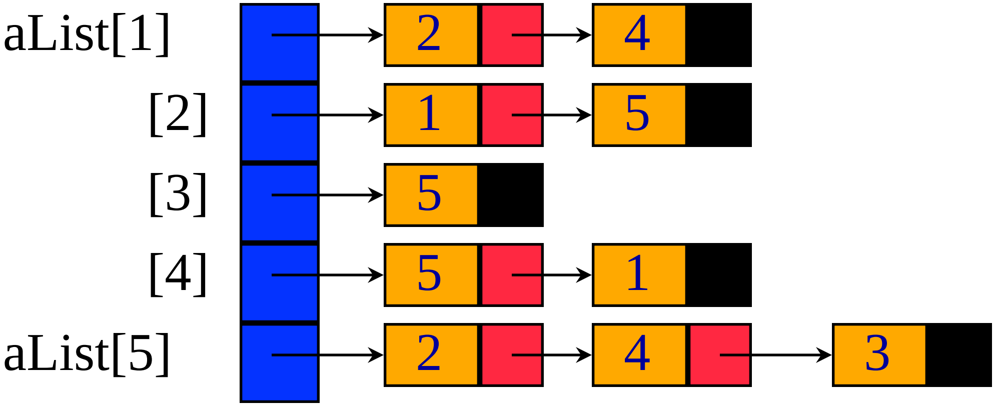
tail	**head**	**column link**	**row link**



Adjacency Multilists

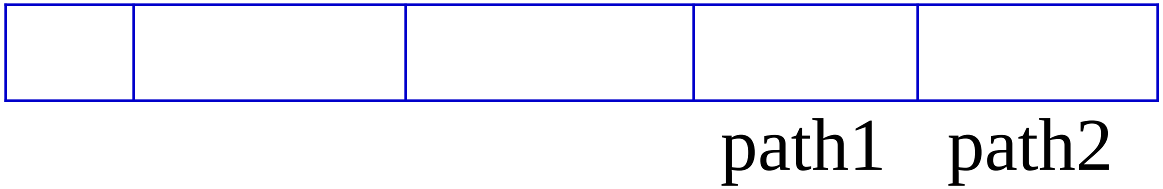
- Undirected graph





Each (u, v) is represented by 2 entries.

Visit an edge only once?



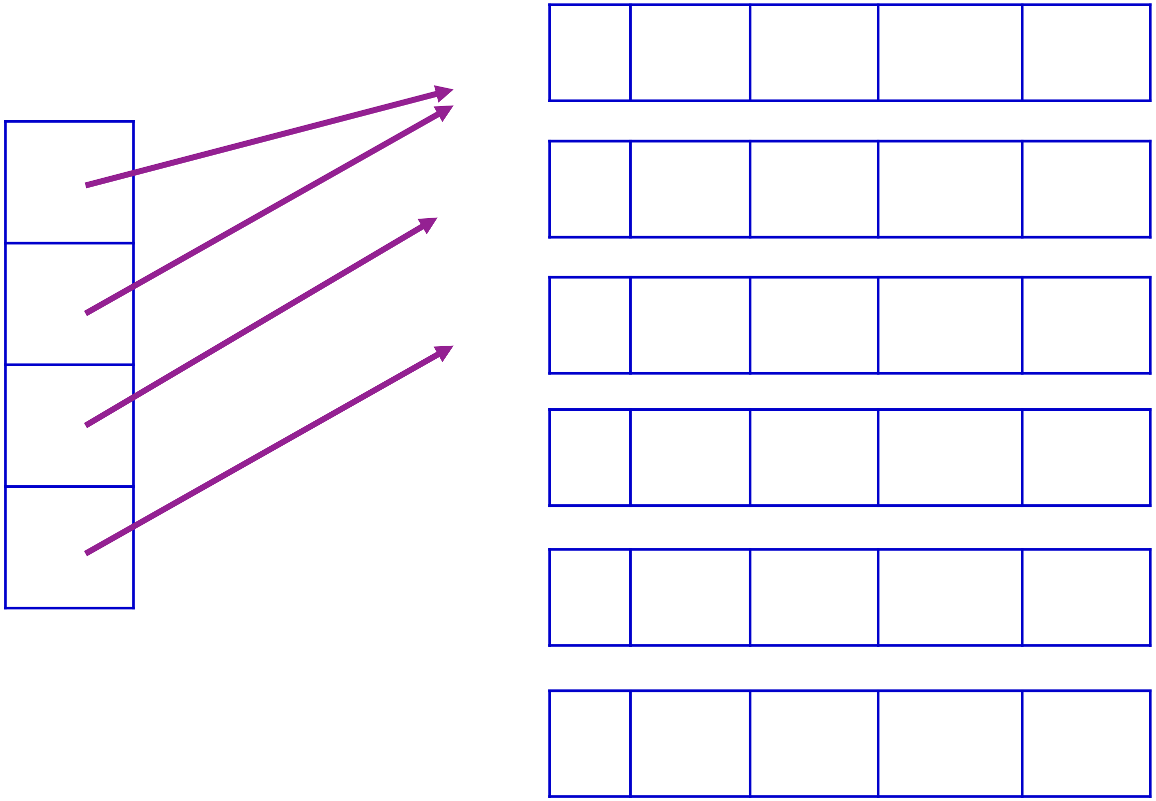
Adjacency Multilists

- **class** MGraphEdge **{**
- **private:**
-     **bool** m**;**
-     **int** vertex1, vertex2**;**
-

Adjacency Multilists

- MGraph::MGraph(**const int** vertices) : e(0)
- **{**
-     **if** (vertices < 1) **throw** “Number of vertices must be > 0”**;**
-     n = vertices**;**
-     adjMultiLists = **new** EdgePtr[n]**;**
-     fill(adjMultiLists, adjMultiLists+n,0)**;**
- **}**

Adjacency Multilists



Adjacency Multilists

- **If p points to an MGraphEdge representing (u, v), and given u, to get v we need the following test:**
    - **if** (p→vertex1 == u) v = p→vertex2**;**
    - **else** v = p→vertex1;
- **And we can insert an edge in O(1):**
- **void** MGraph::InsertEdge(**int** u, **int** v) **{**
-     MGraphEdge *p = **new** MGraphEdge**;**
-     p→m = **false;** p→vertex1 = u**;** p→vertex2 = v;
-     p→path1 = adjMultiLists[u]**;**
-     p→path2 = adjMultiLists[v]**;**
-     adjMultiLists[u] = adhMultiLists[v] = p;
- }

Weighted Graphs

- Cost adjacency matrix.
  - C(i,j) = cost of edge (i,j)
- Adjacency lists => each list element is a pair (adjacent vertex, edge weight)

Number Of Classes Needed

- Graph representations
  - Adjacency Matrix
  - Adjacency Lists
    - Linked Adjacency Lists
    - Array Adjacency Lists
  - 3 representations
- Graph types
  - Directed and undirected.
  - Weighted and unweighted.
  - 2 x 2 = 4 graph types
- 3 x 4 = 12 classes

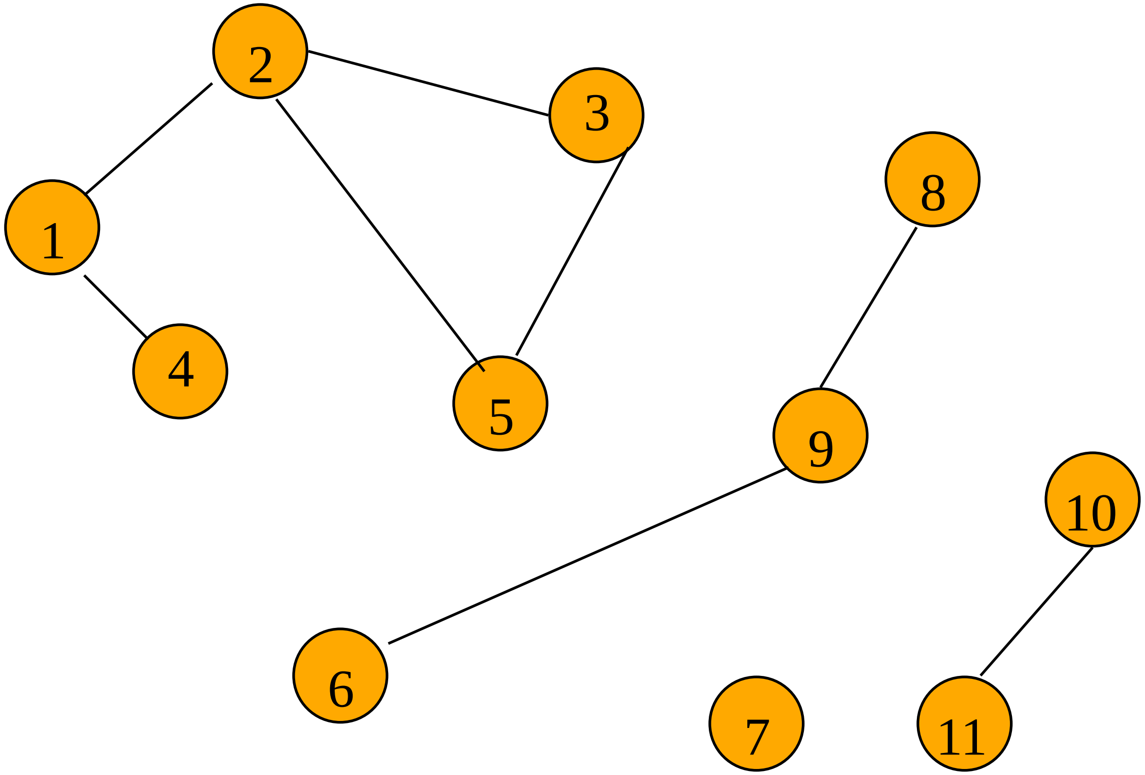
P340-5, 9

Graph Search Methods

- Given G = (V, E), and v in V(G), we wish to visit all vertices in G that are reachable from v.
- In the following methods, we assume the graphs are undirected, although they work on the directed as well.

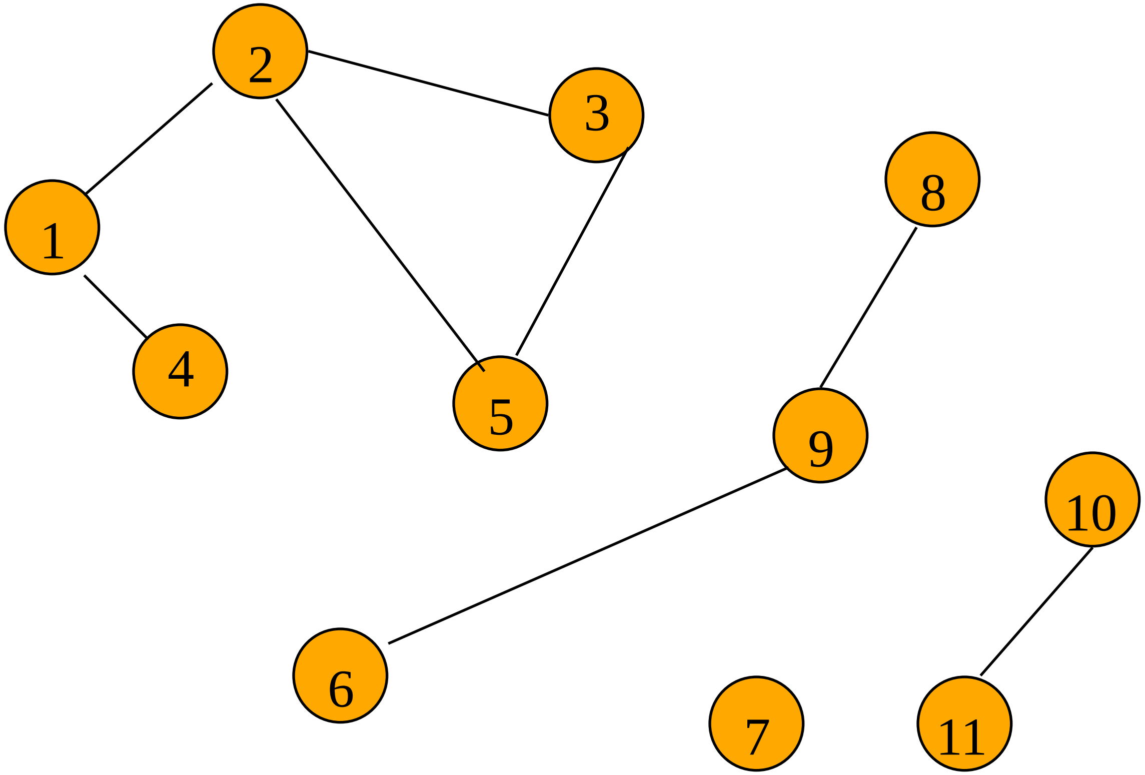
Graph Search Methods

- A vertex u is reachable from vertex v iff there is a path from v to u.



Graph Search Methods

- A search method starts at a given vertex v and visits/labels/marks every vertex that is reachable from v.



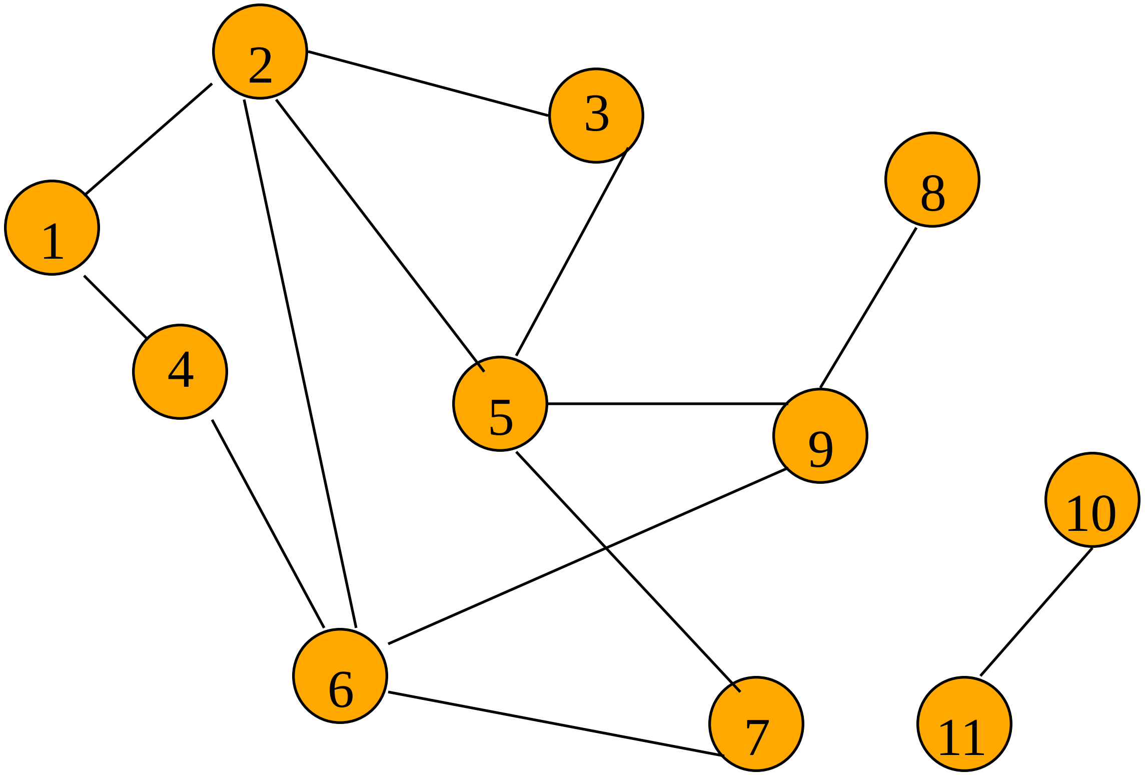
Graph Search Methods

- Many graph problems solved using a search method.
  - Path from one vertex to another.
  - Is the graph connected?
  - Find a spanning tree.
  - Etc.
- Commonly used search methods:
  - Breadth-first search.
  - Depth-first search.

Breadth-First Search

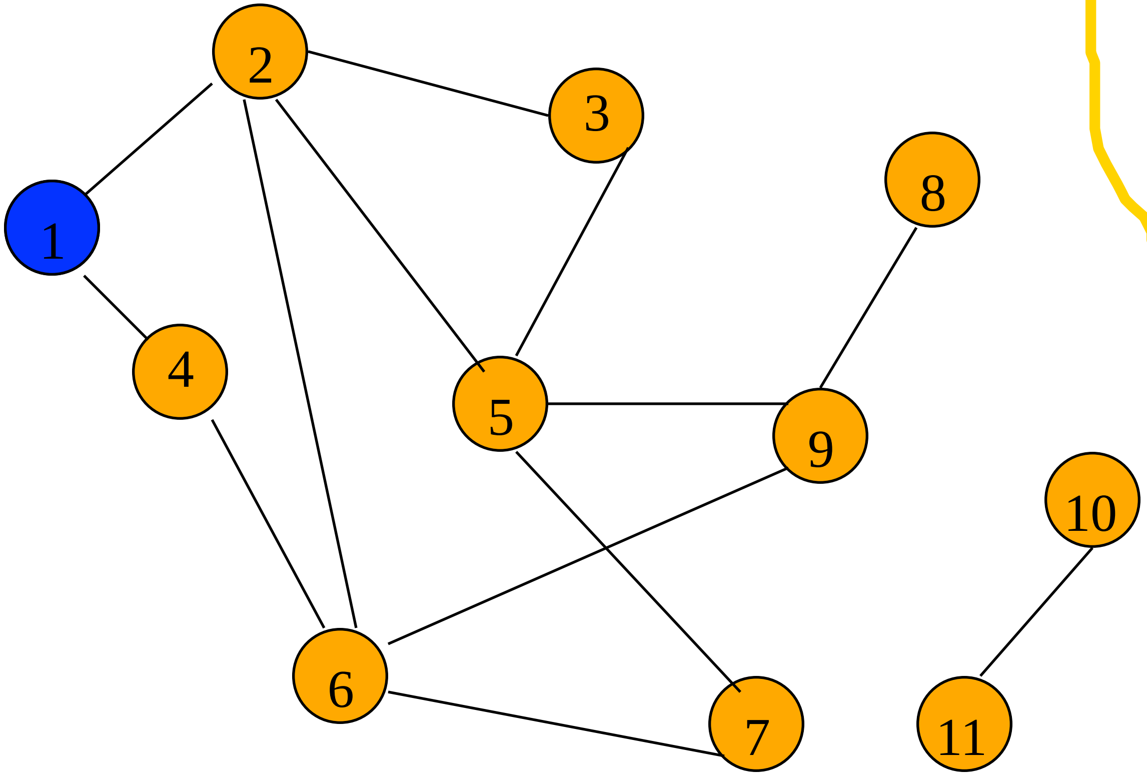
- Visit start vertex and put into a FIFO queue.
- Repeatedly remove a vertex from the queue, visit its unvisited adjacent vertices, put newly visited vertices into the queue.

Breadth-First Search Example



Start search at vertex 1.

Breadth-First Search Example

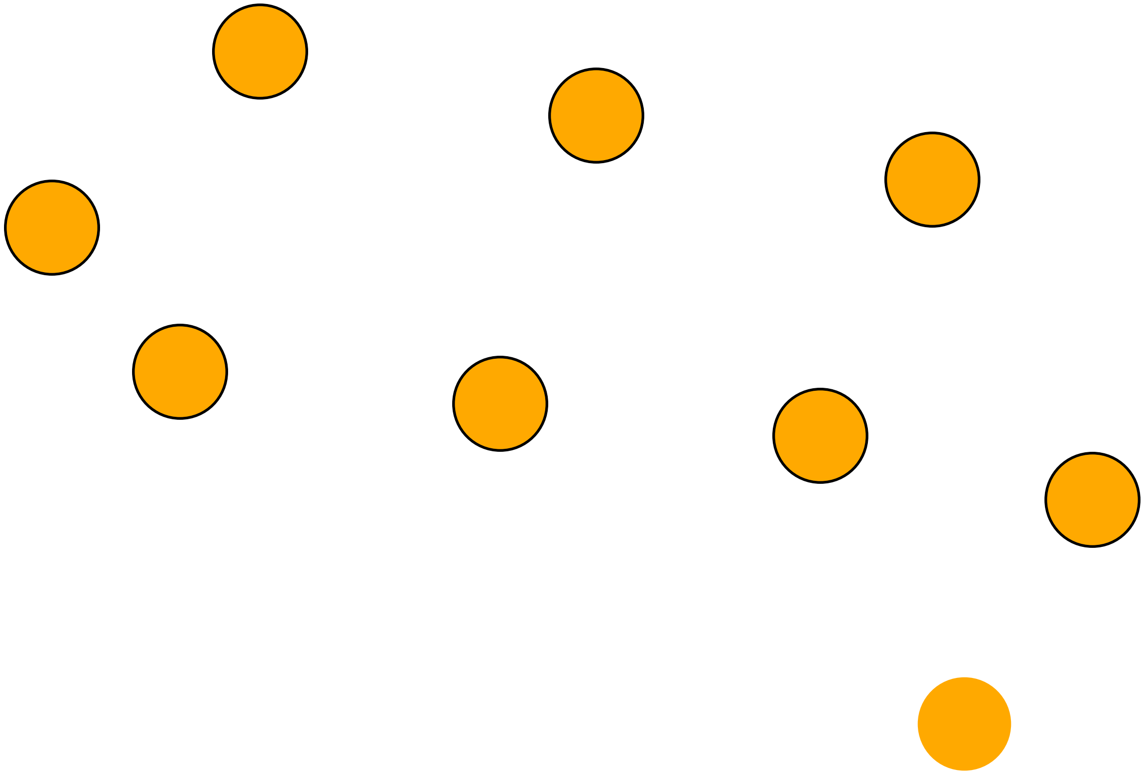


FIFO Queue

1

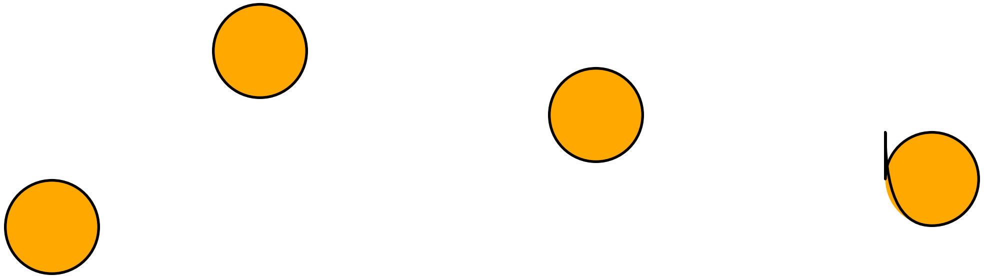
Visit/mark/label start vertex and put in a FIFO queue.

Breadth-First Search Example



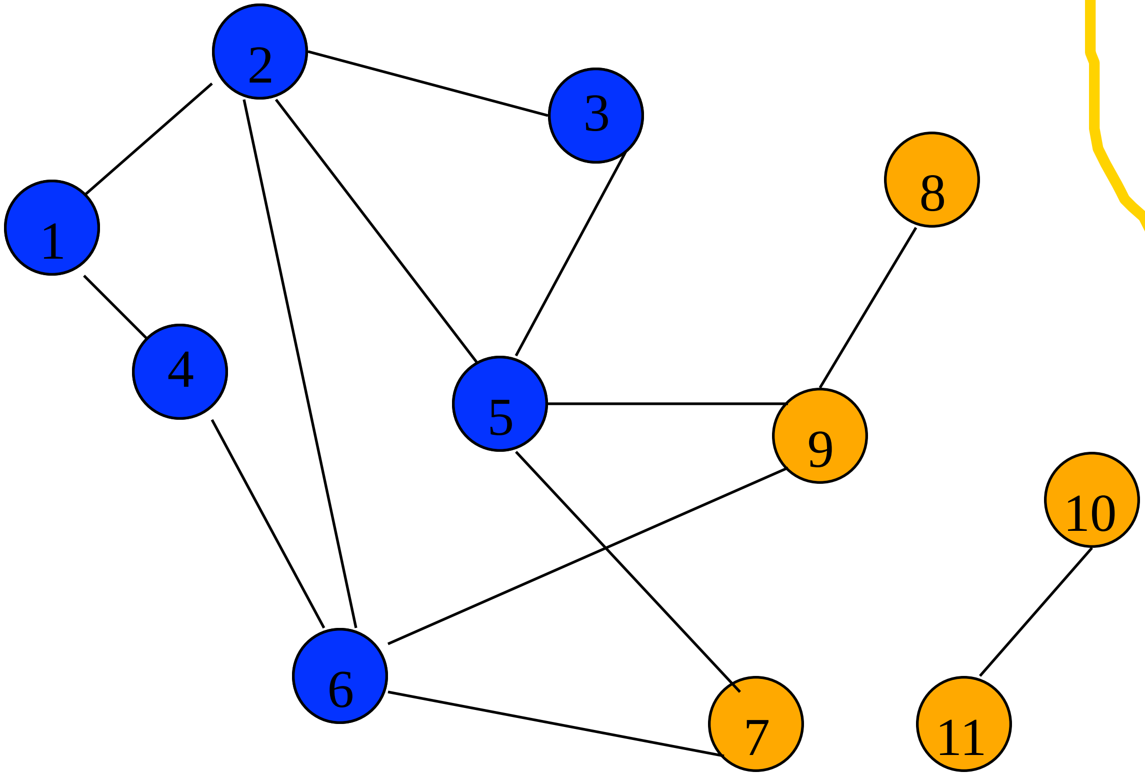
Remove 1 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example



Remove 2 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example

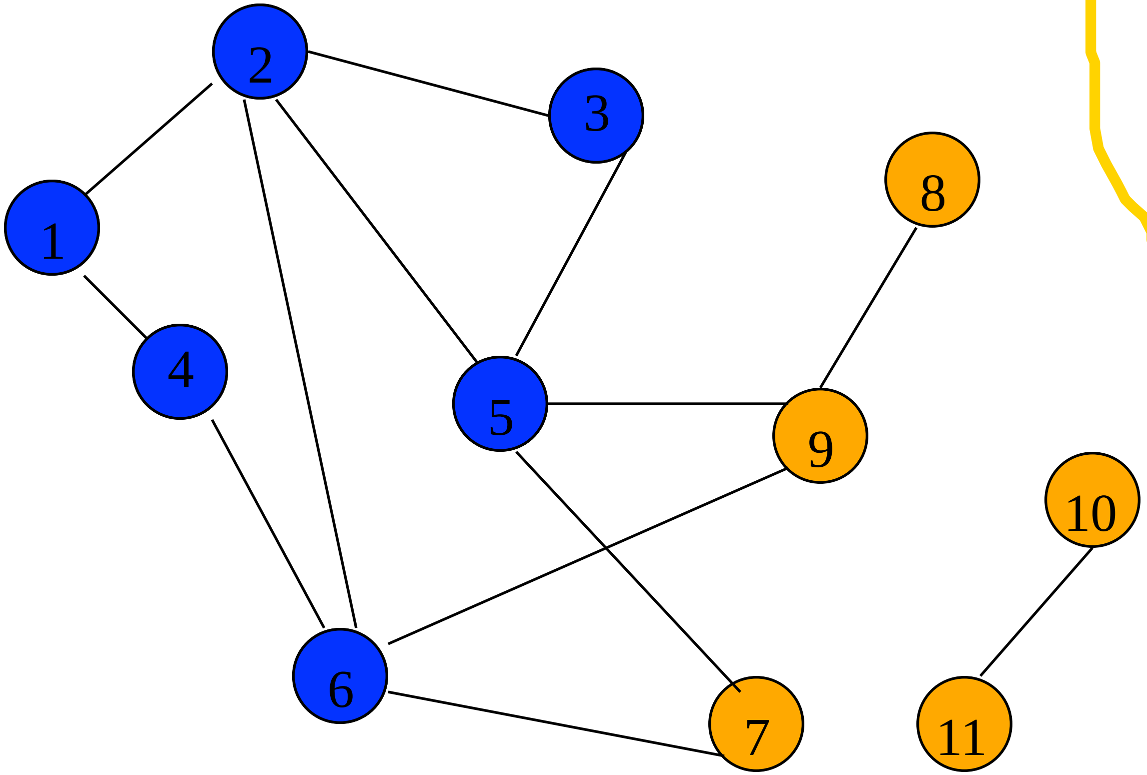


FIFO Queue

4 5 3 6

Remove 4 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example

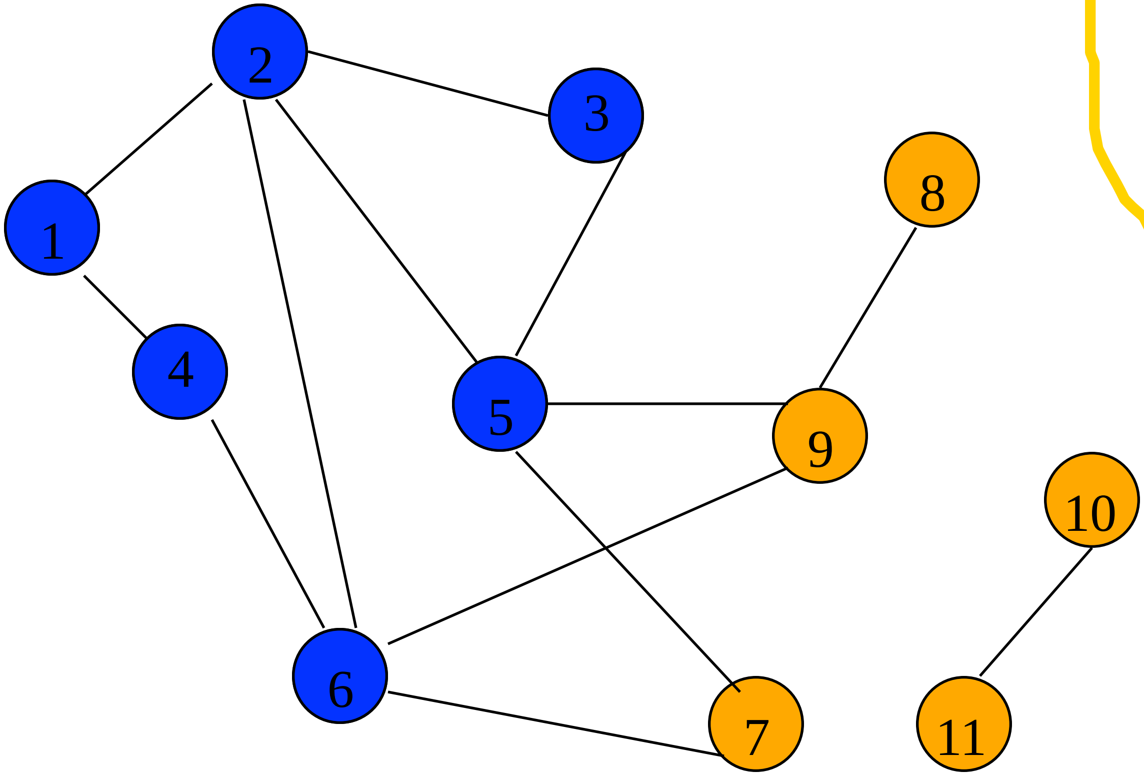


FIFO Queue

5 3 6

Remove 4 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example

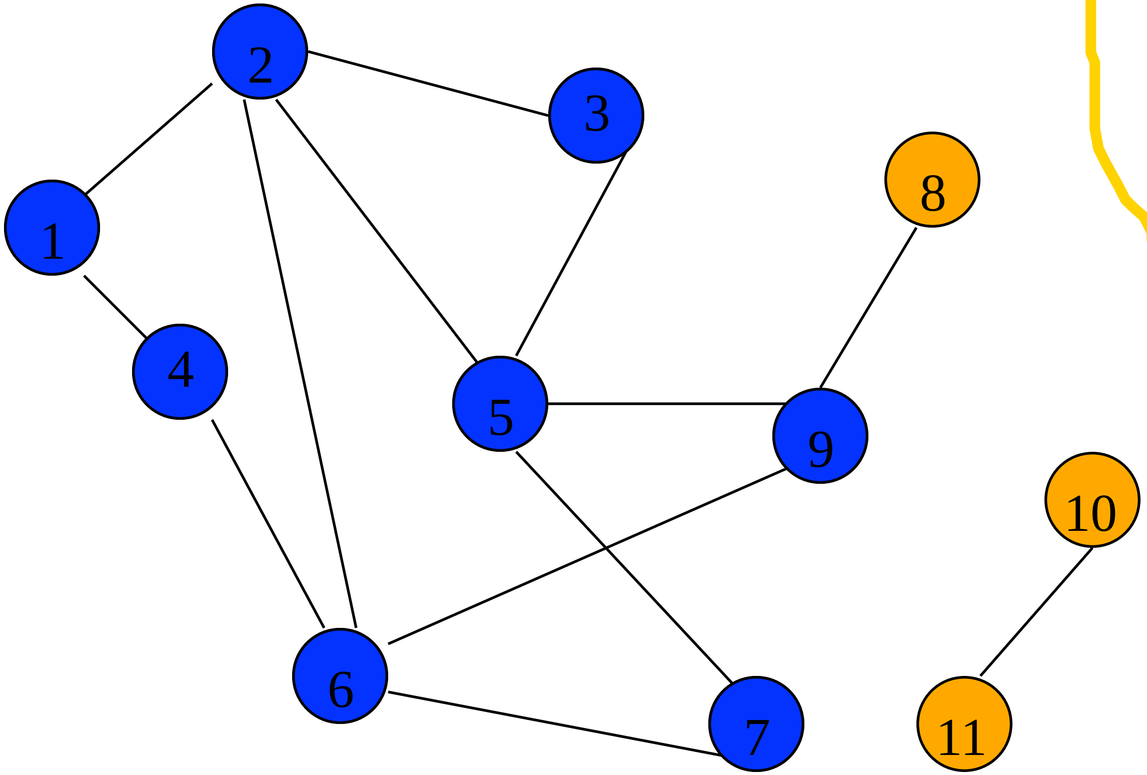


FIFO Queue

5 3 6

Remove 5 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example

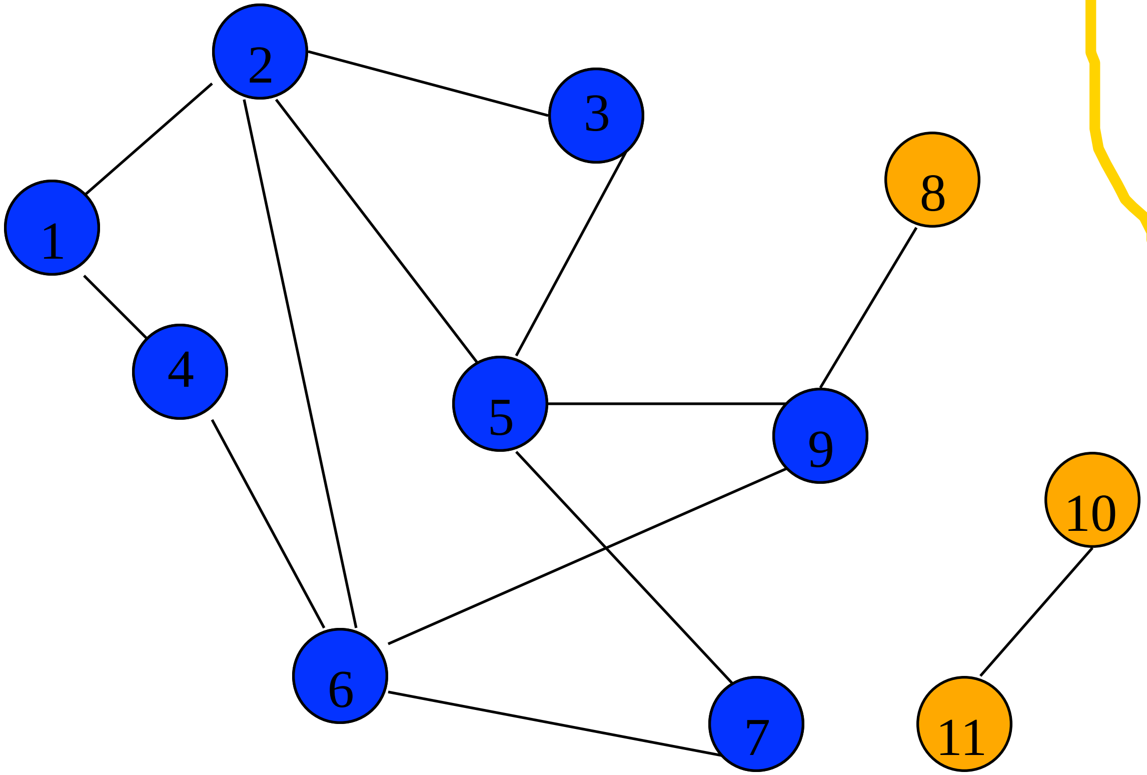


FIFO Queue

3 6 9 7

Remove 5 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example

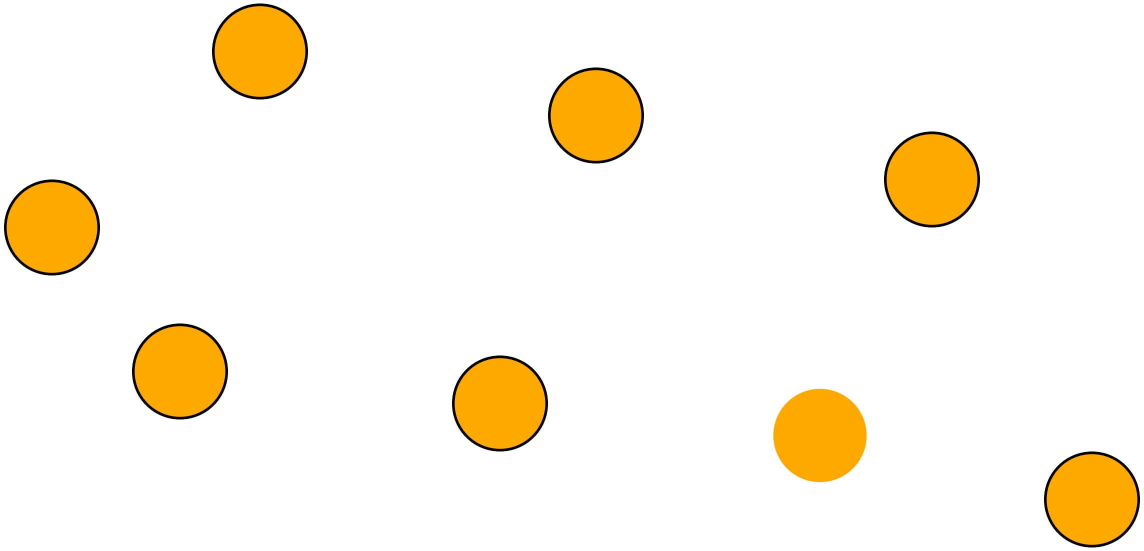


FIFO Queue

3 6 9 7

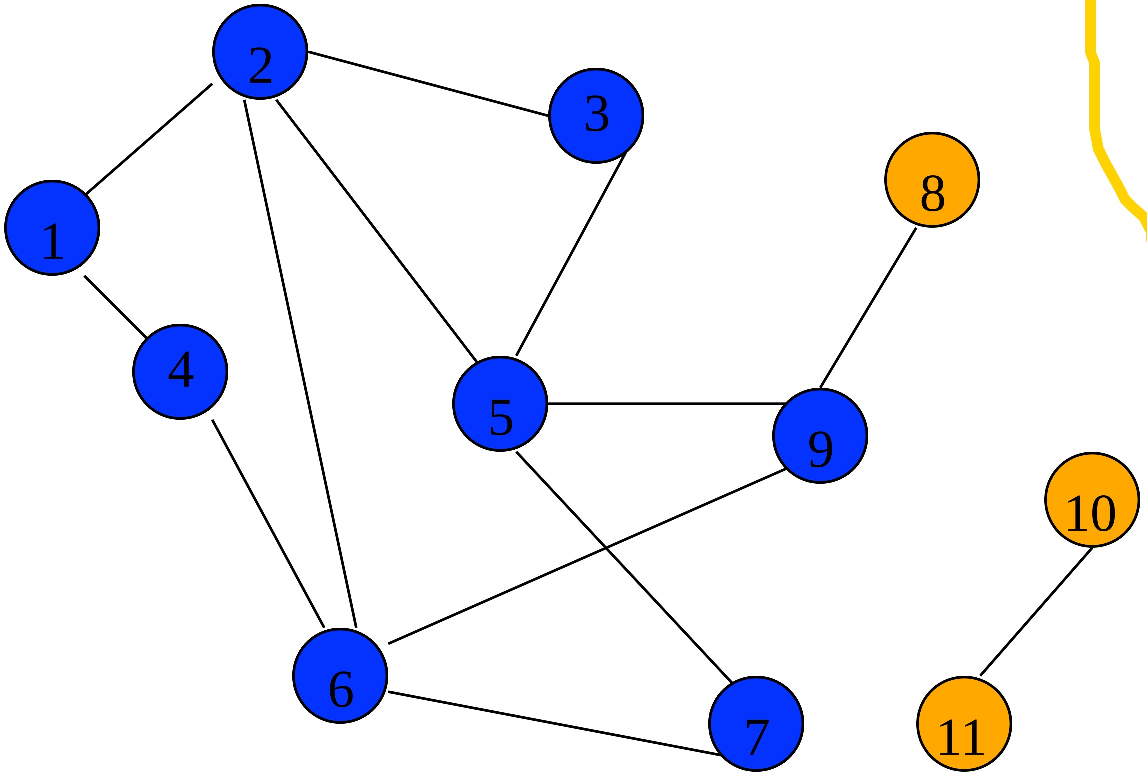
Remove 3 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example



Remove 3 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example

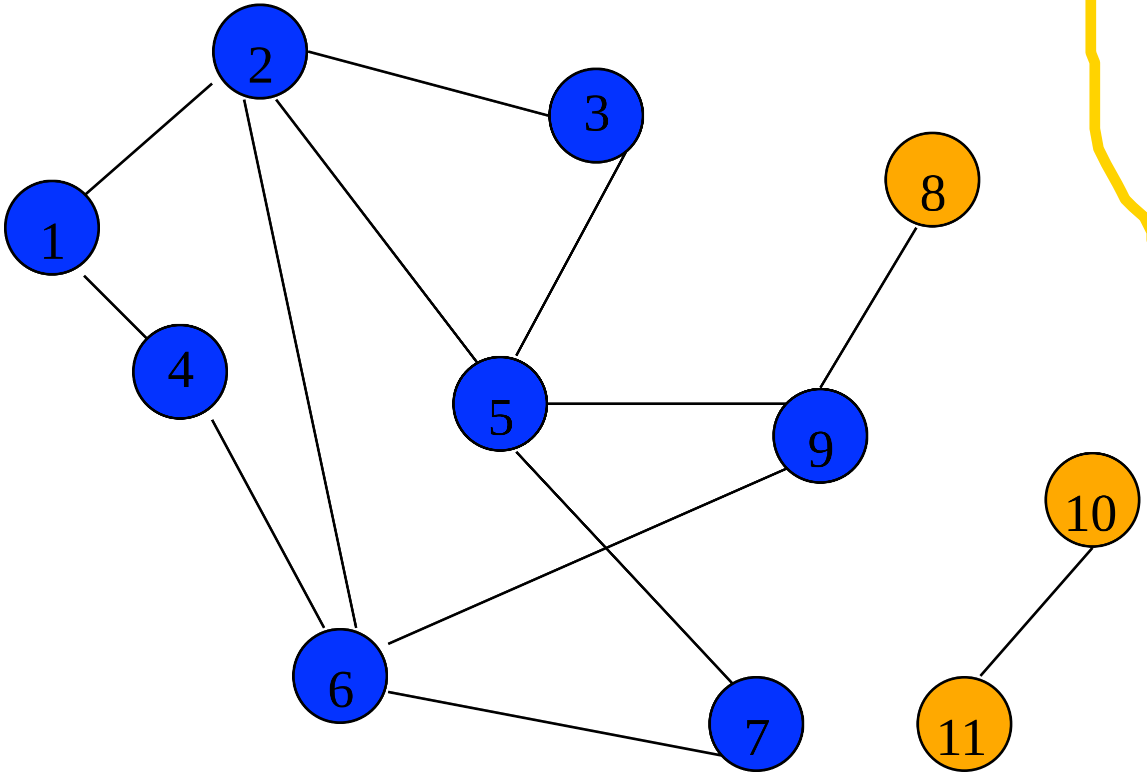


FIFO Queue

6 9 7

Remove 6 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example

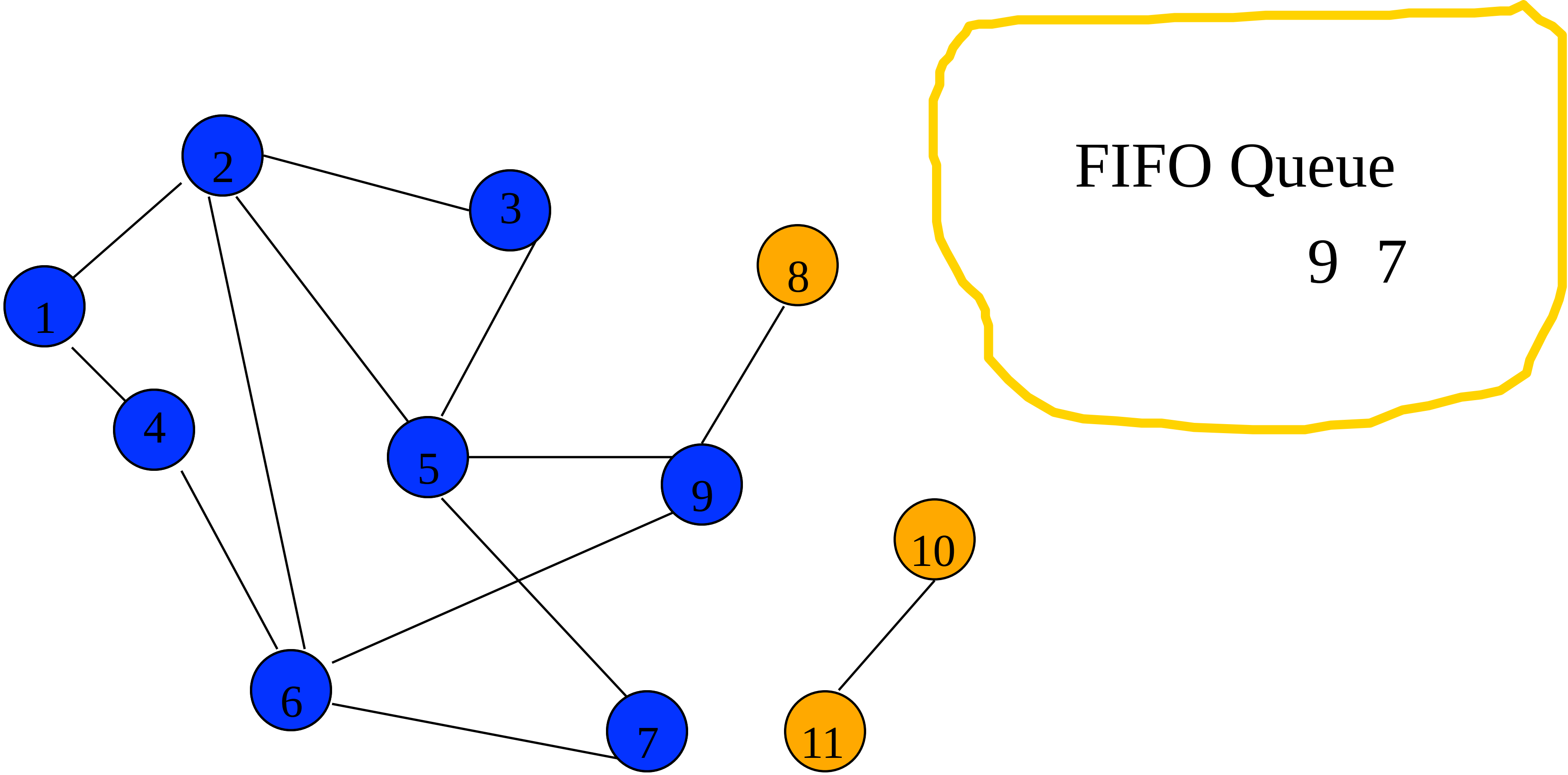


FIFO Queue

9 7

Remove 6 from Q; visit adjacent unvisited vertices; put in Q.

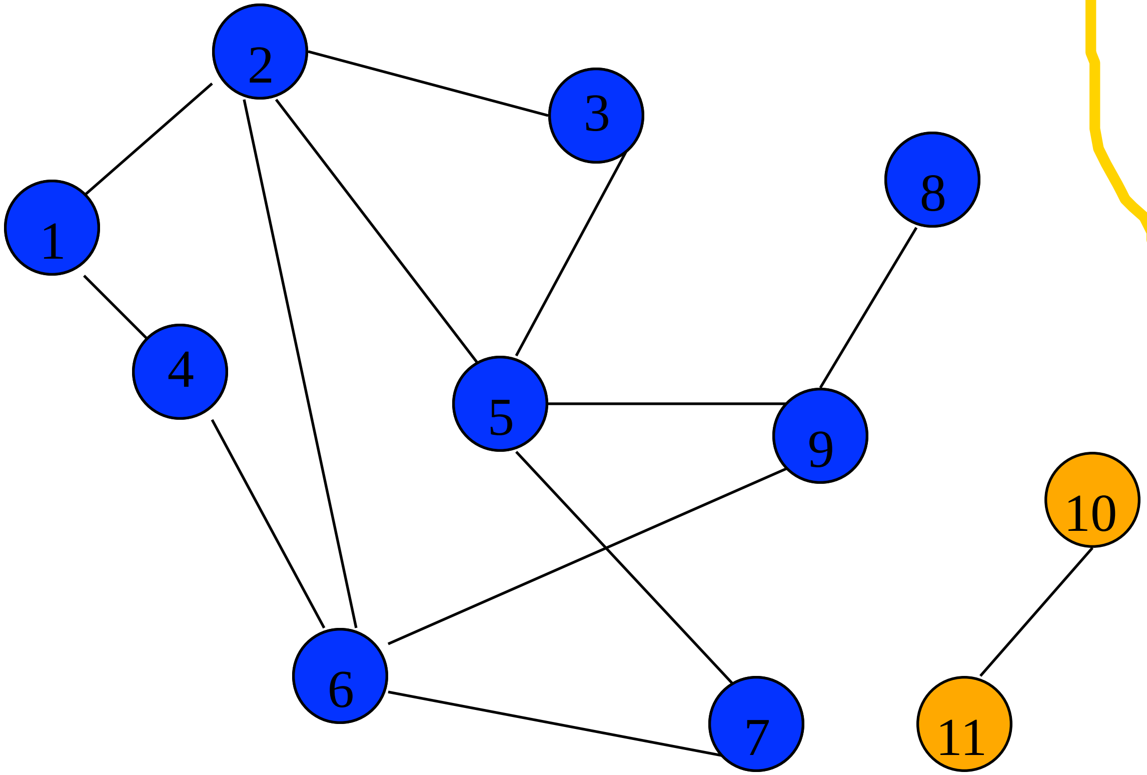
Breadth-First Search Example



Remove 9 from Q; visit adjacent unvisited vertices;

put in Q.

Breadth-First Search Example

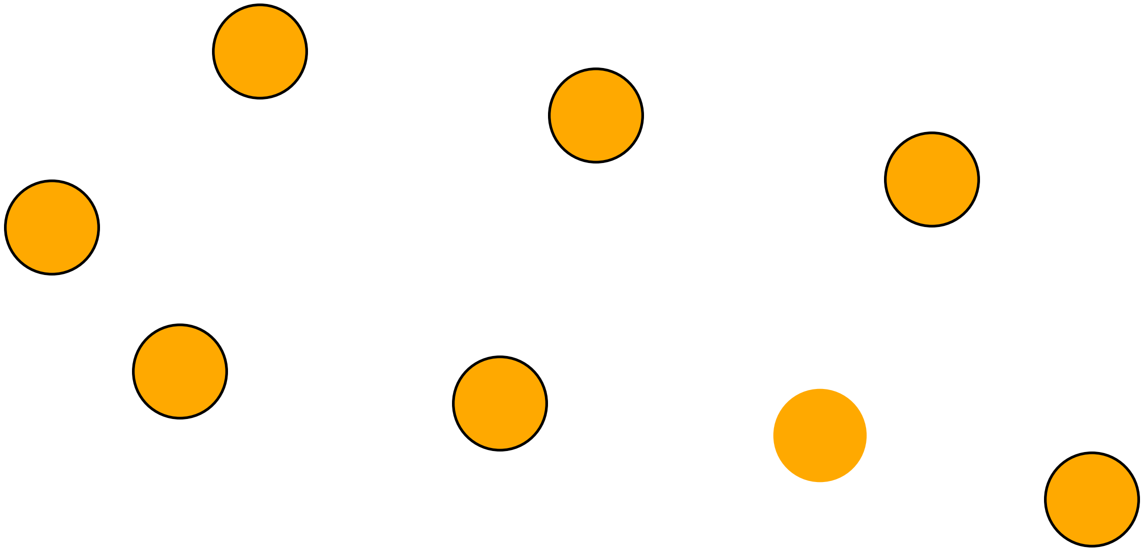


FIFO Queue

7   8

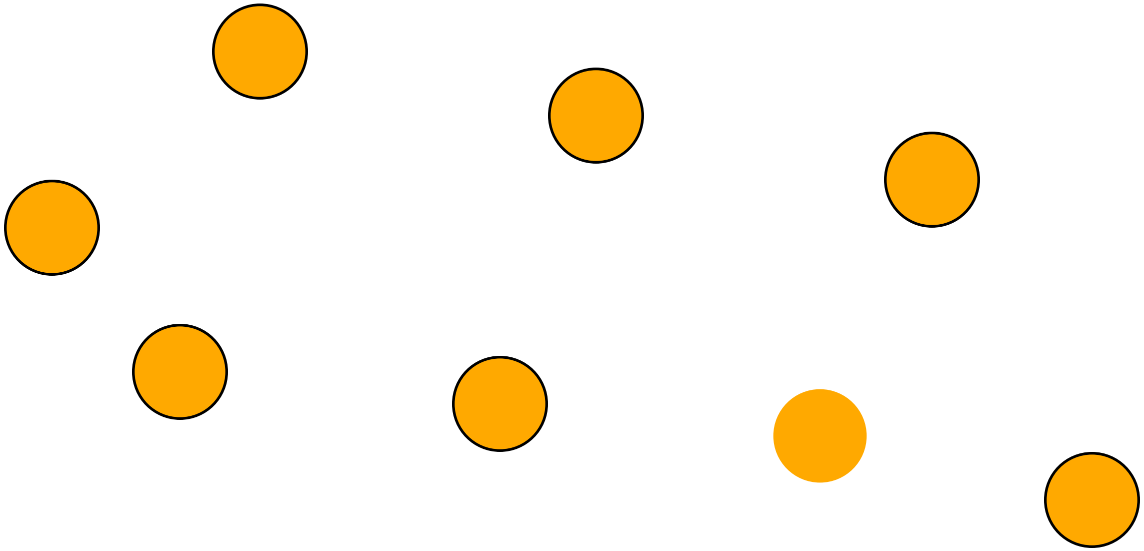
Remove 9 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example



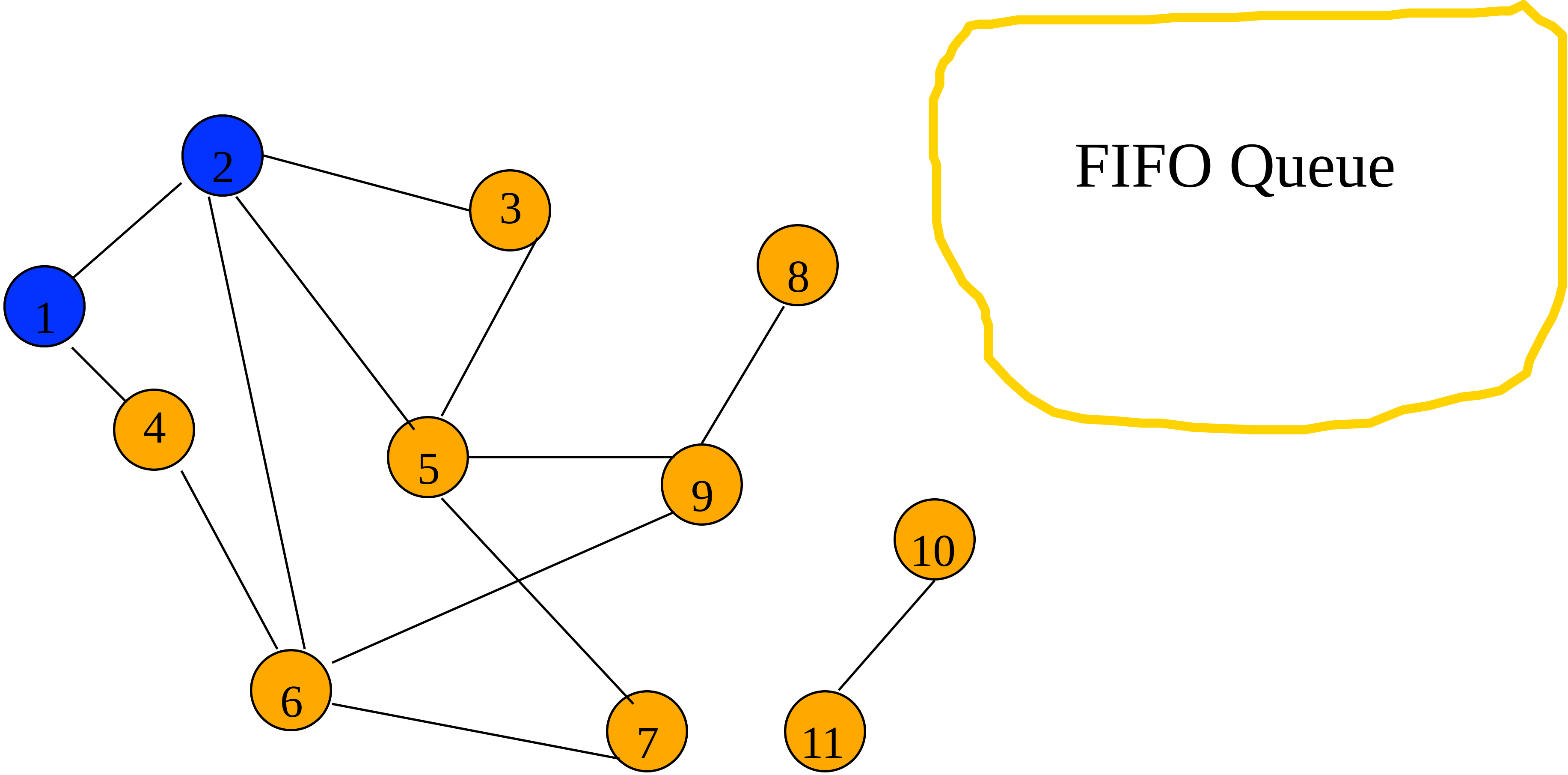
Remove 7 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example



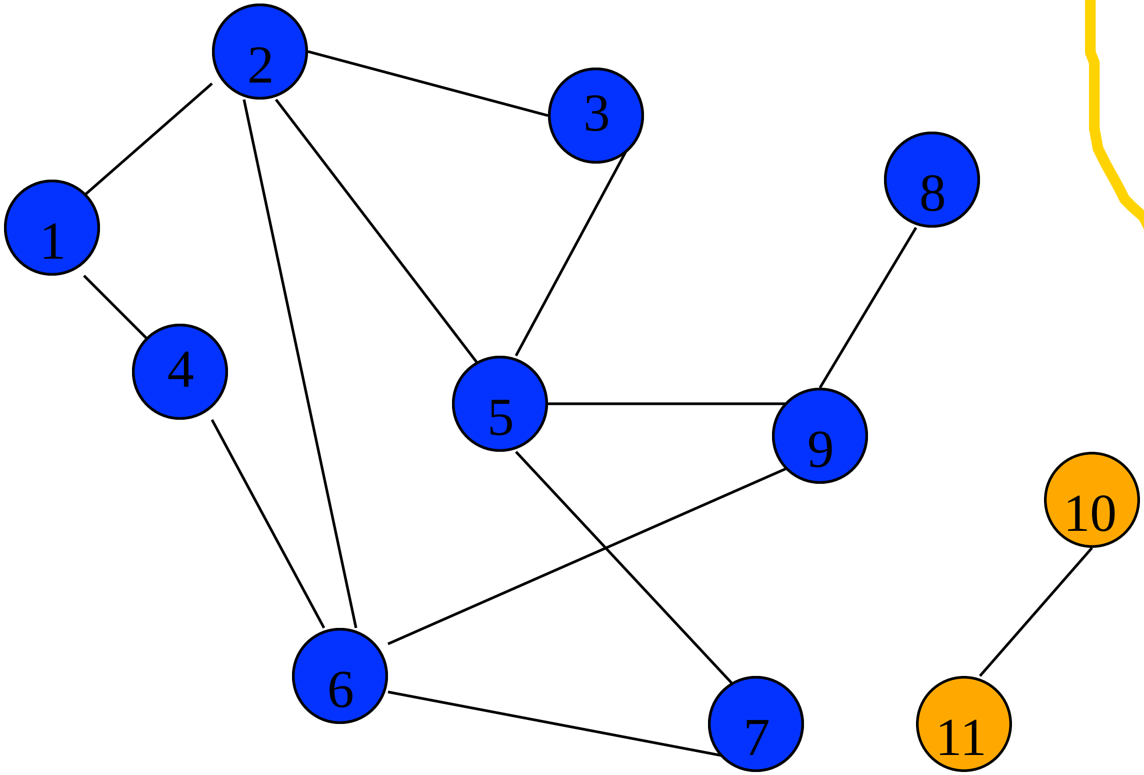
Remove 7 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example



Remove 8 from Q; visit adjacent unvisited vertices; put in Q.

Breadth-First Search Example



FIFO Queue

Queue is empty. Search terminates.

Breadth-First Search Property

- All vertices reachable from the start vertex (including the start vertex) are visited.

```
• virtual void Graph::BFS (int v) {
•     visited = new bool[n];    fill(visited, visited + n, false);
•     visited[v] = true;
•     Queue<int> q;
•     q.Push(v);
•     while ( !q.IsEmpty()) {
•          v = q.Front();    q.Pop();
•         for (all vertices w adjacent to v)
•            if (!visited[w]) {
•                 visited[w] = true;
•                 q.Push(w);
•
•   }           end of while
•     delete [ ] visited;
• }
```

Time Complexity

- Each visited vertex is put on (and so removed from) the queue exactly once.
- When a vertex is removed from the queue, we examine its adjacent vertices.
  - O(n) if adjacency matrix used
  - O(vertex degree) if adjacency lists used
- Total time
  - O(mn), where m is number of vertices in the component that is searched (adjacency matrix)

Time Complexity

- O(n + sum of component vertex degrees) (adj. lists)

  = O(n + number of edges in component)

Path From Vertex v To Vertex u

- Start a breadth-first search at vertex v.
- Terminate when vertex u is visited or when Q becomes empty (whichever occurs first).
- Time
  - $O(n^2)$ when adjacency matrix used
  - $O(n+e)$ when adjacency lists used (e is number of edges)

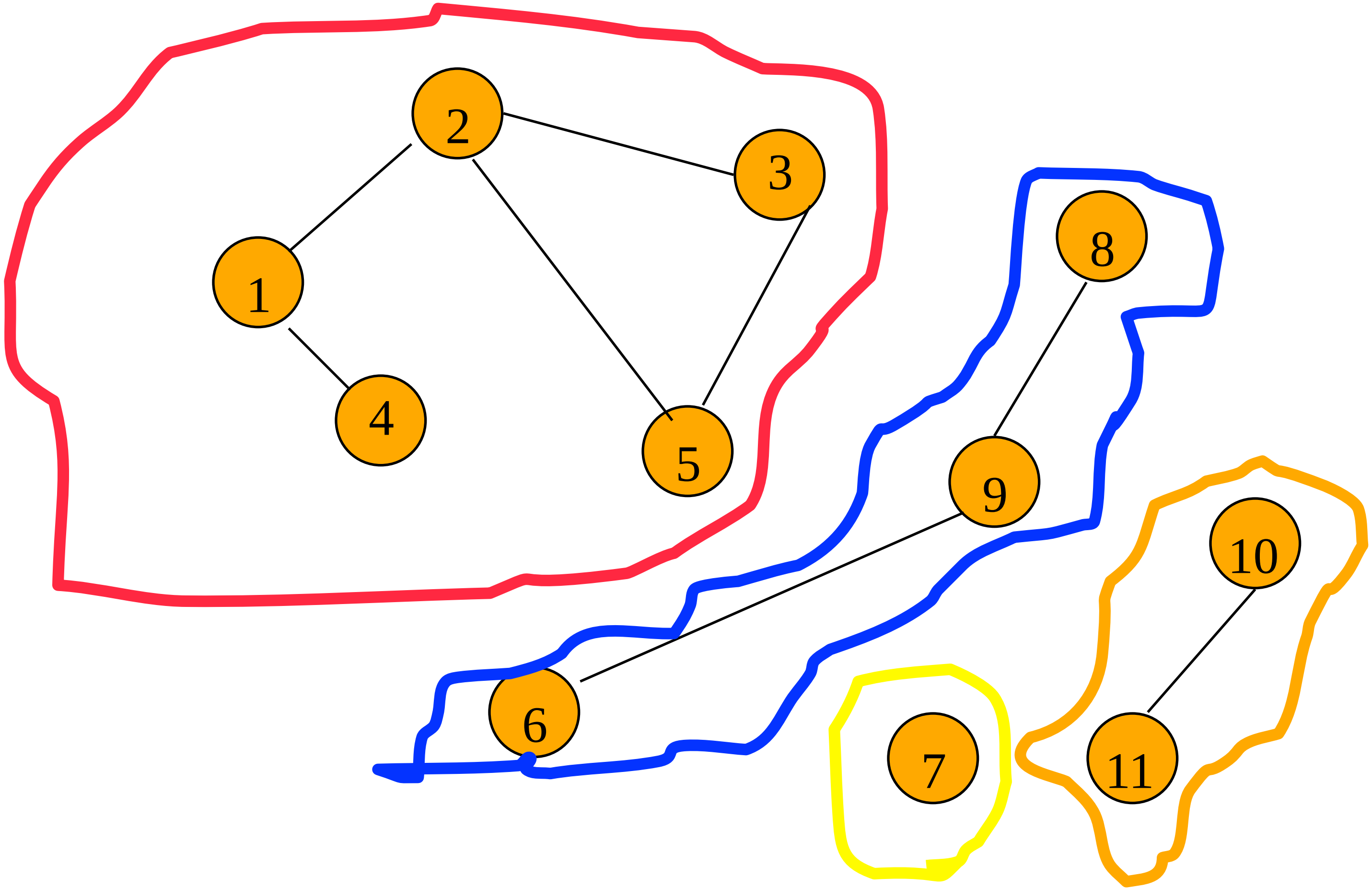
Is The Graph Connected?

- Start a breadth-first search at any vertex of the graph.
- Graph is connected iff all n vertices get visited.
- Time
  - $O(n^2)$ when adjacency matrix used
  - $O(n+e)$ when adjacency lists used (e is number of edges)

Connected Components

- Start a breadth-first search at any as yet unvisited vertex of the graph.
- Newly visited vertices (plus edges between them) define a component.
- Repeat until all vertices are visited.

Connected Components



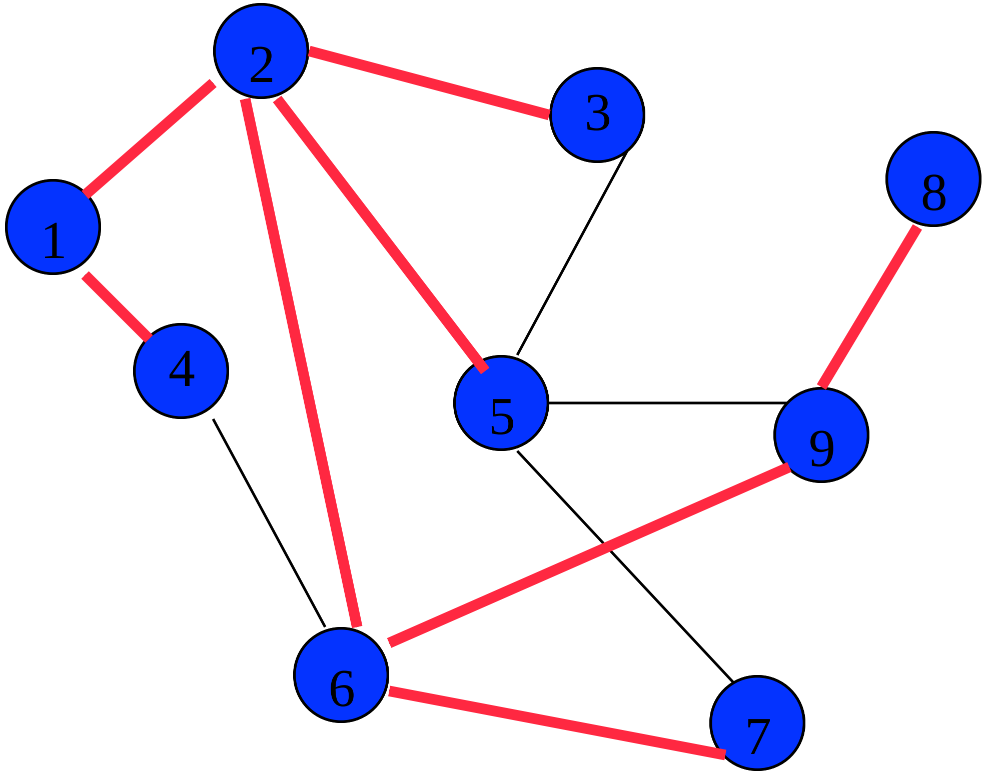
Connected Components

```
virtual void Graph::Components(){
    visited = new bool[n];
    fill(visited, visited+n, false);
    for (int i=0; i<n; i++)
        if (!visited[i]) {
            BFS (i); // find a component
             OutputNewComponent();
               }
    delete [ ] visited;
}
```

Time Complexity

- $O(n^2)$ when adjacency matrix used
- $O(n+e)$ when adjacency lists used (e is number of edges)

Spanning Tree



Breadth-first search from vertex 1.

Breadth-first spanning tree.

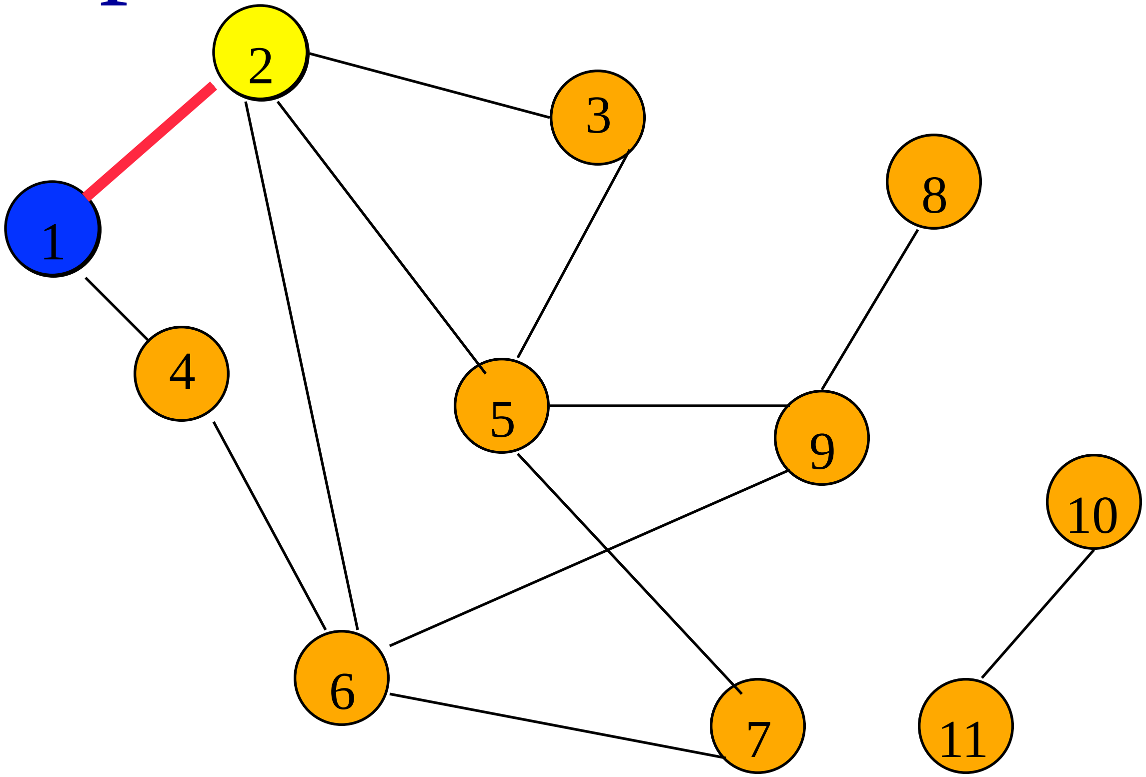
Spanning Tree

- Start a breadth-first search at any vertex of the graph.
- If graph is connected, the n-1 edges used to get to unvisited vertices define a spanning tree (breadth-first spanning tree).
- Time
  - $O(n^2)$ when adjacency matrix used
  - O(n+e) when adjacency lists used (e is number of edges)

Depth-First Search

```
depthFirstSearch(v)
{
  Label vertex v as reached.
  for (each unreached vertex u
                         adjacenct from v)
    depthFirstSearch(u);
}
```

Depth-First Search Example

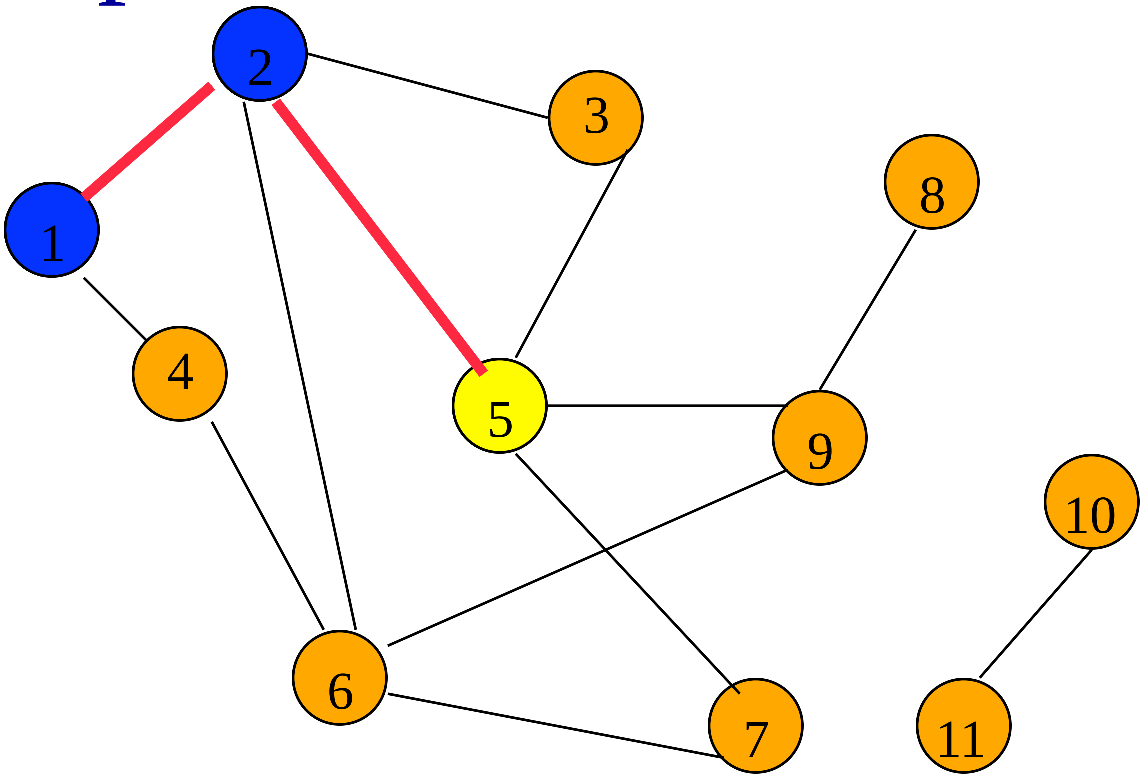


Start search at vertex 1.

Label vertex 1 and do a depth first search from either 2 or 4.

Suppose that vertex 2 is selected.

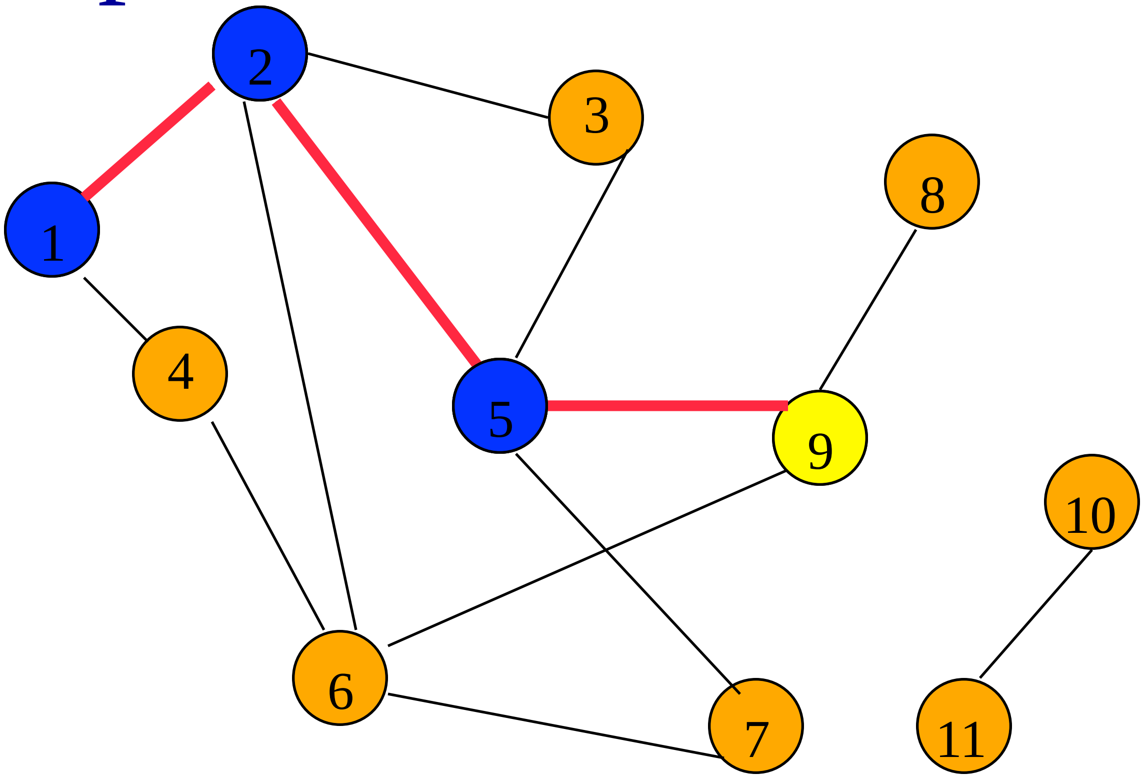
Depth-First Search Example



Label vertex 2 and do a depth first search from either 3, 5, or 6.

Suppose that vertex 5 is selected.

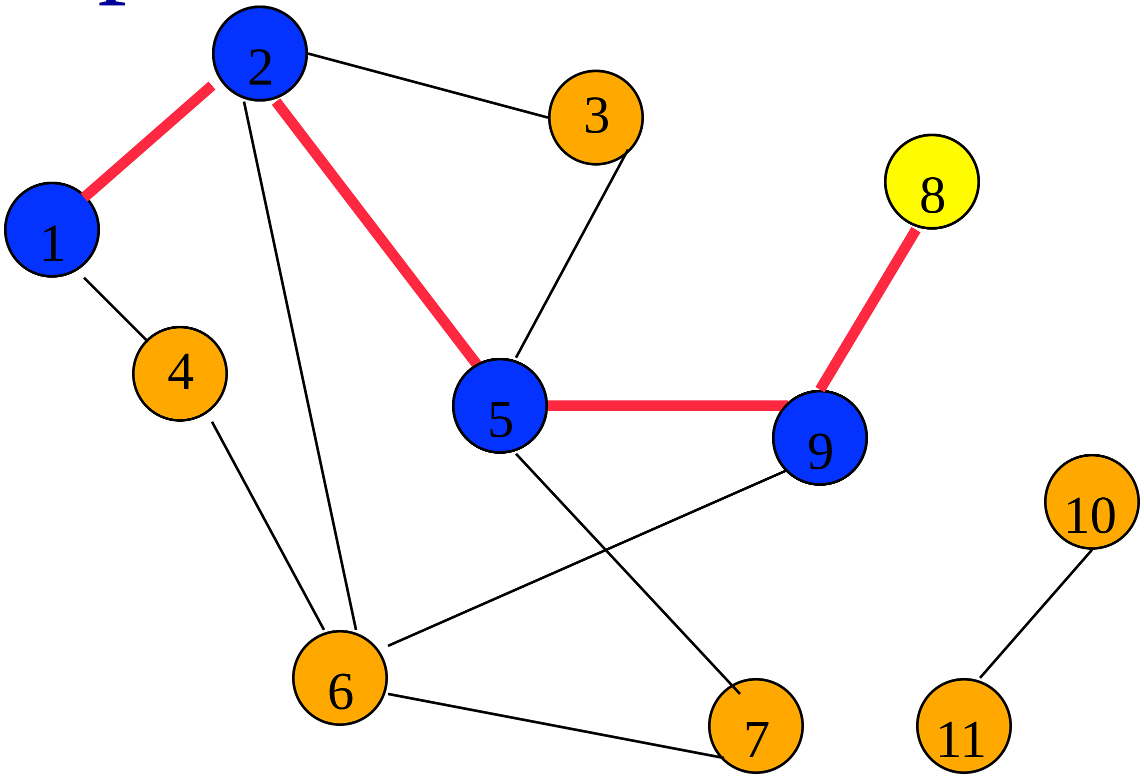
Depth-First Search Example



Label vertex 5 and do a depth first search from either 3, 7, or 9.

Suppose that vertex 9 is selected.

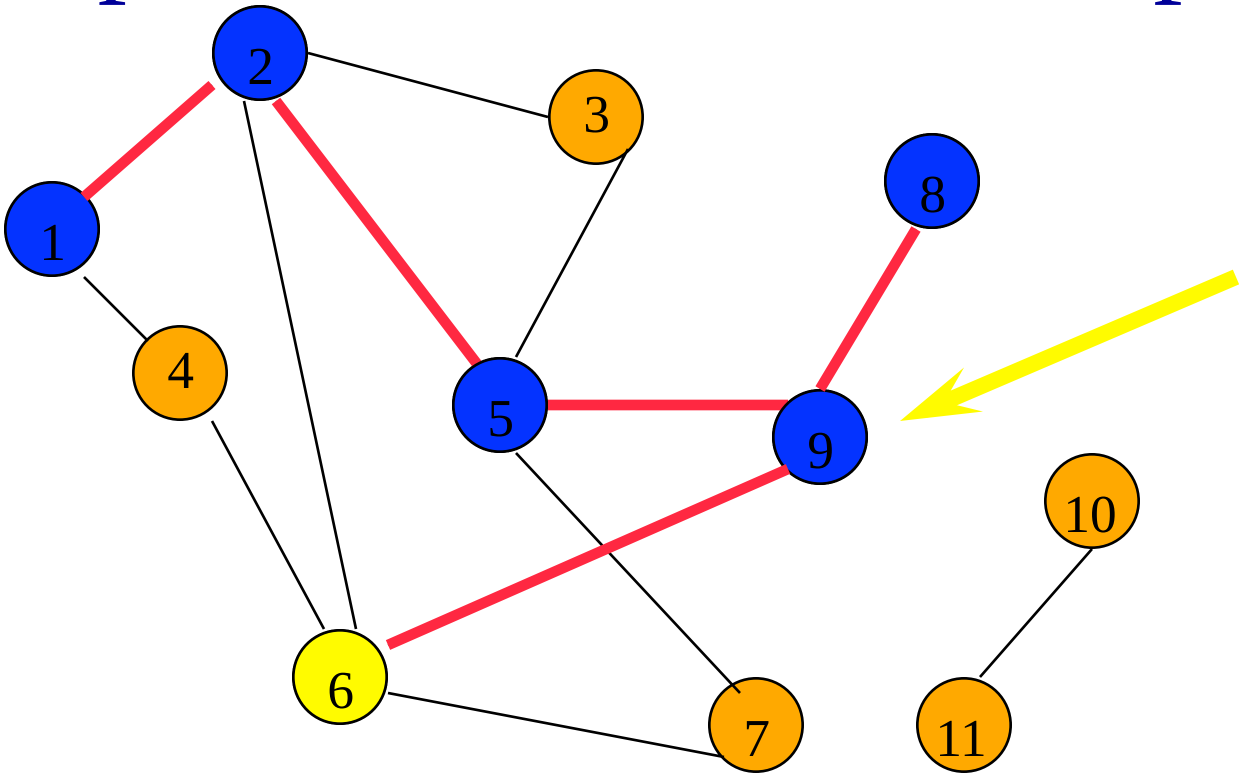
Depth-First Search Example



Label vertex 9 and do a depth first search from either 6 or 8.

Suppose that vertex 8 is selected.

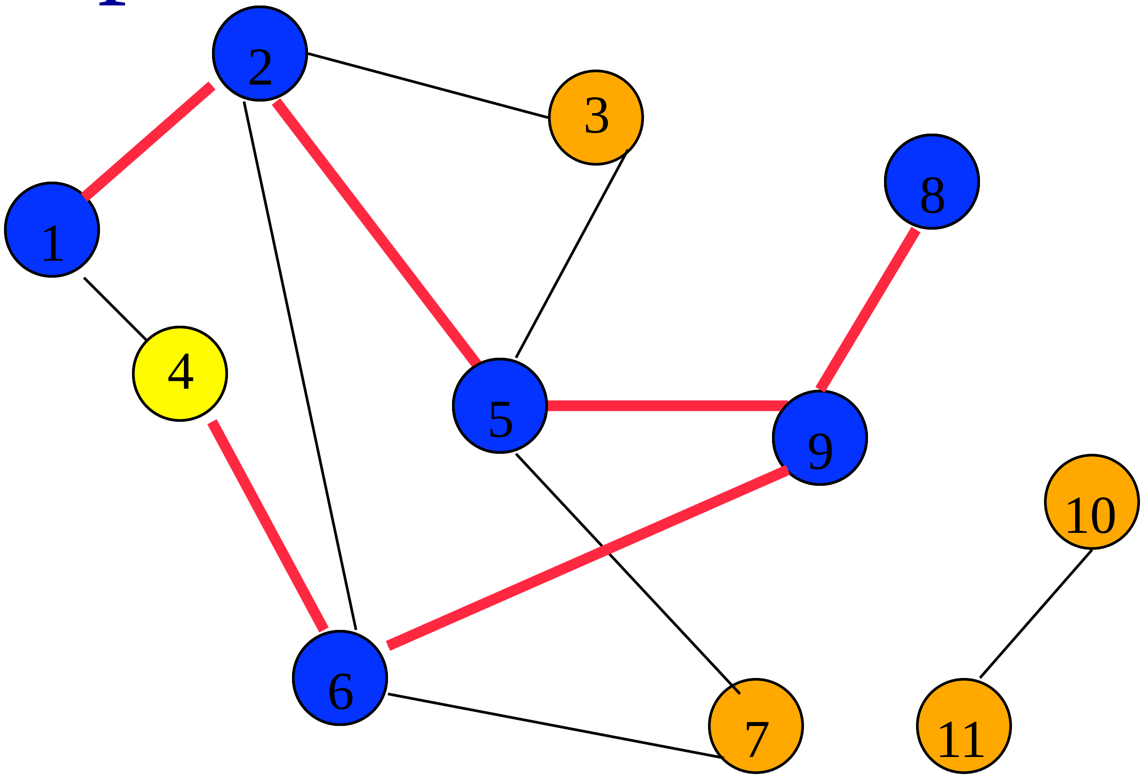
Depth-First Search Example



Label vertex 8 and return to vertex 9.

From vertex 9 do a dfs(6).

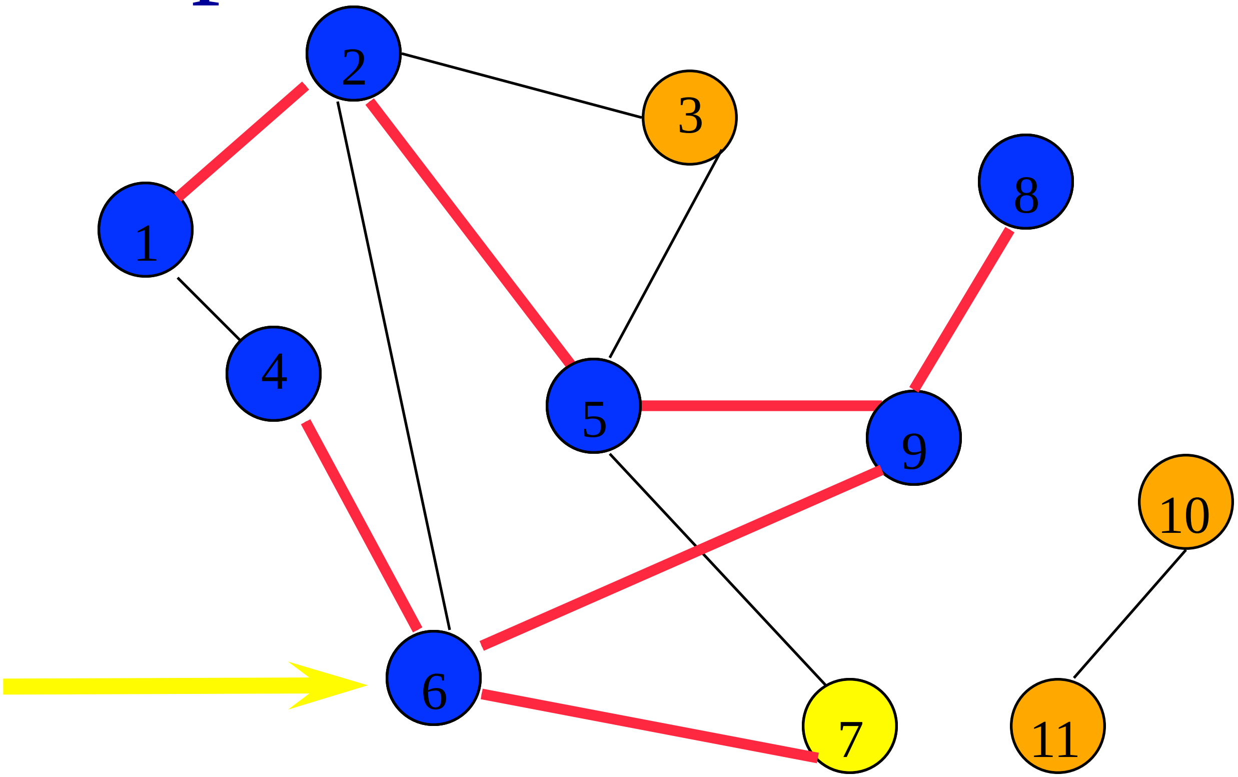
Depth-First Search Example



Label vertex 6 and do a depth first search from either 4 or 7.

Suppose that vertex 4 is selected.

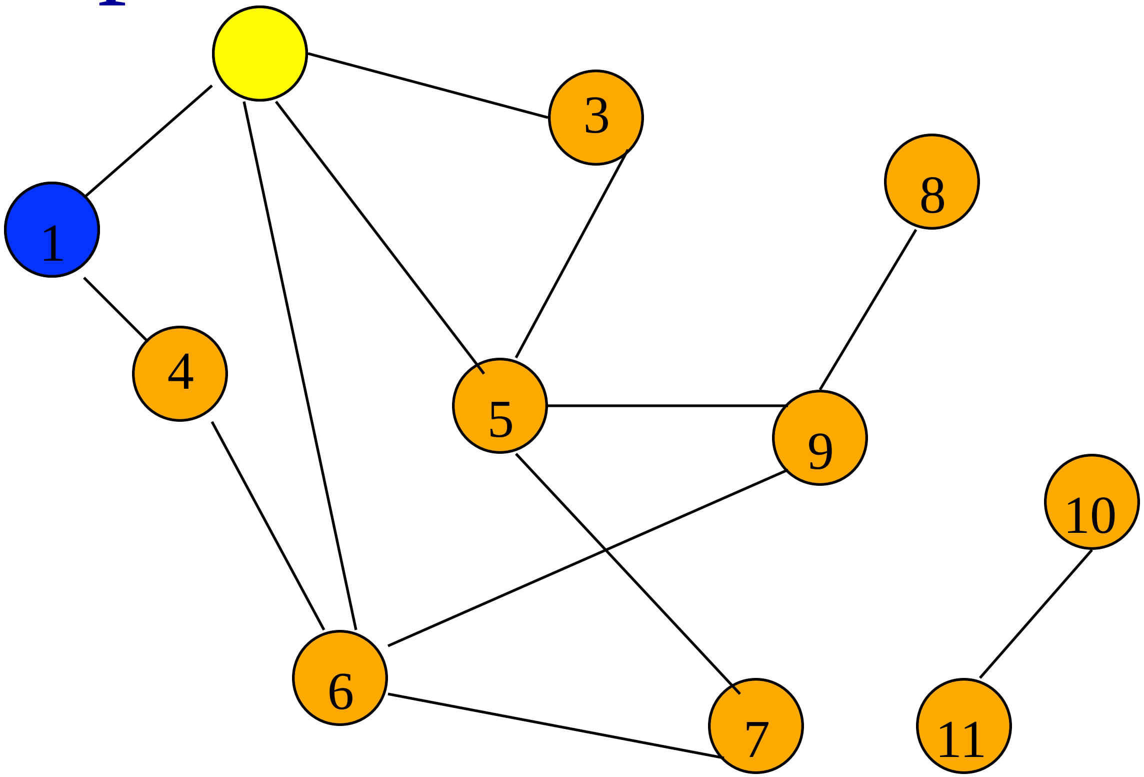
Depth-First Search Example



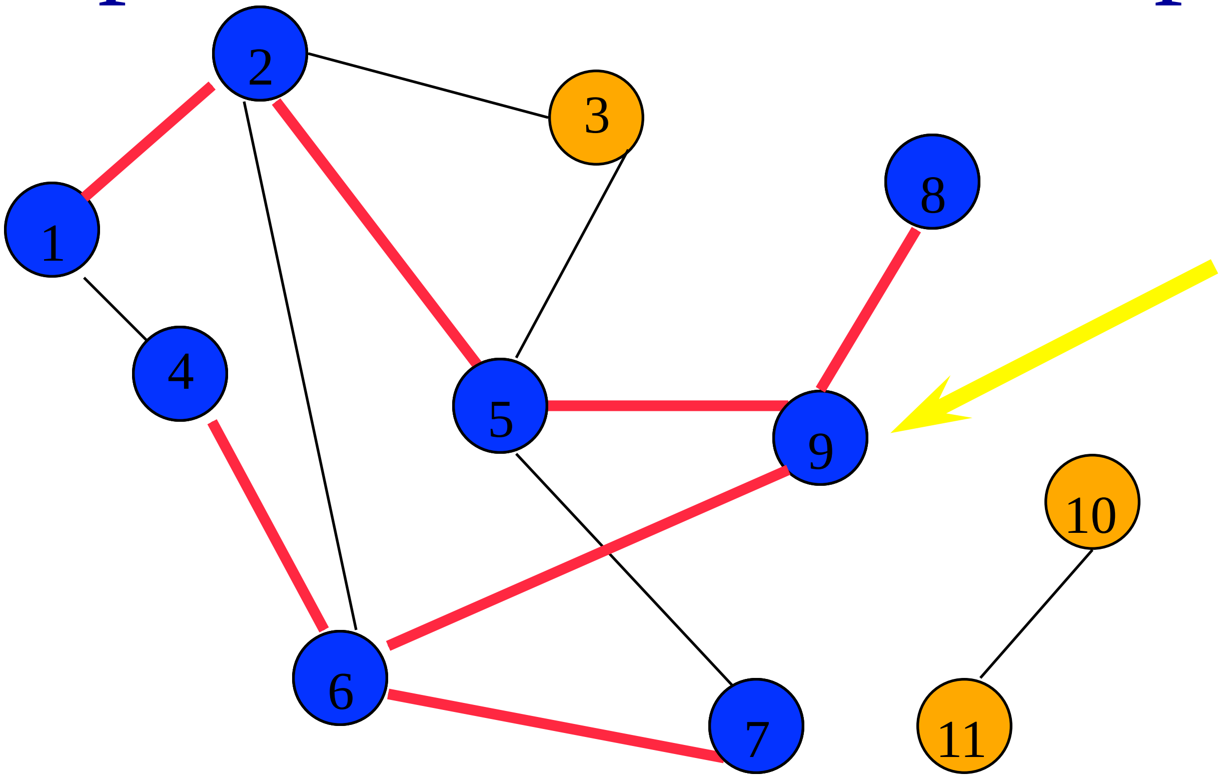
Label vertex 4 and return to 6.

From vertex 6 do a dfs(7).

Depth-First Search Example

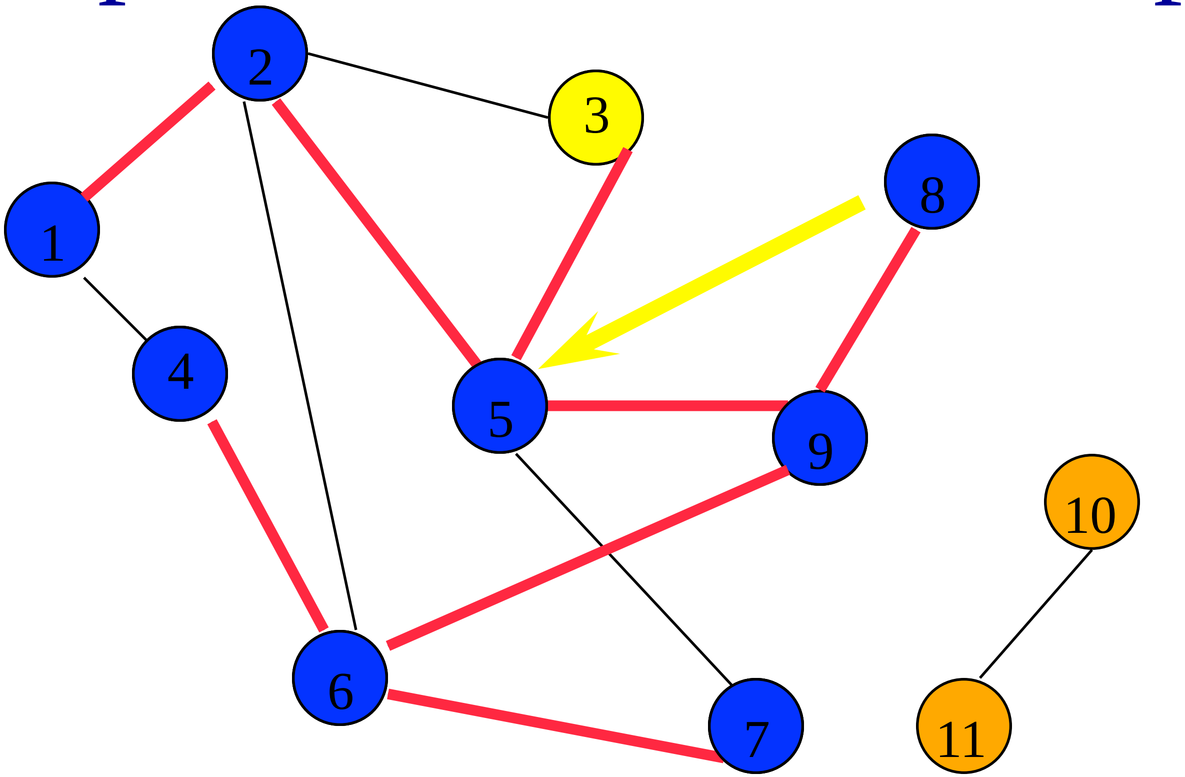


Depth-First Search Example



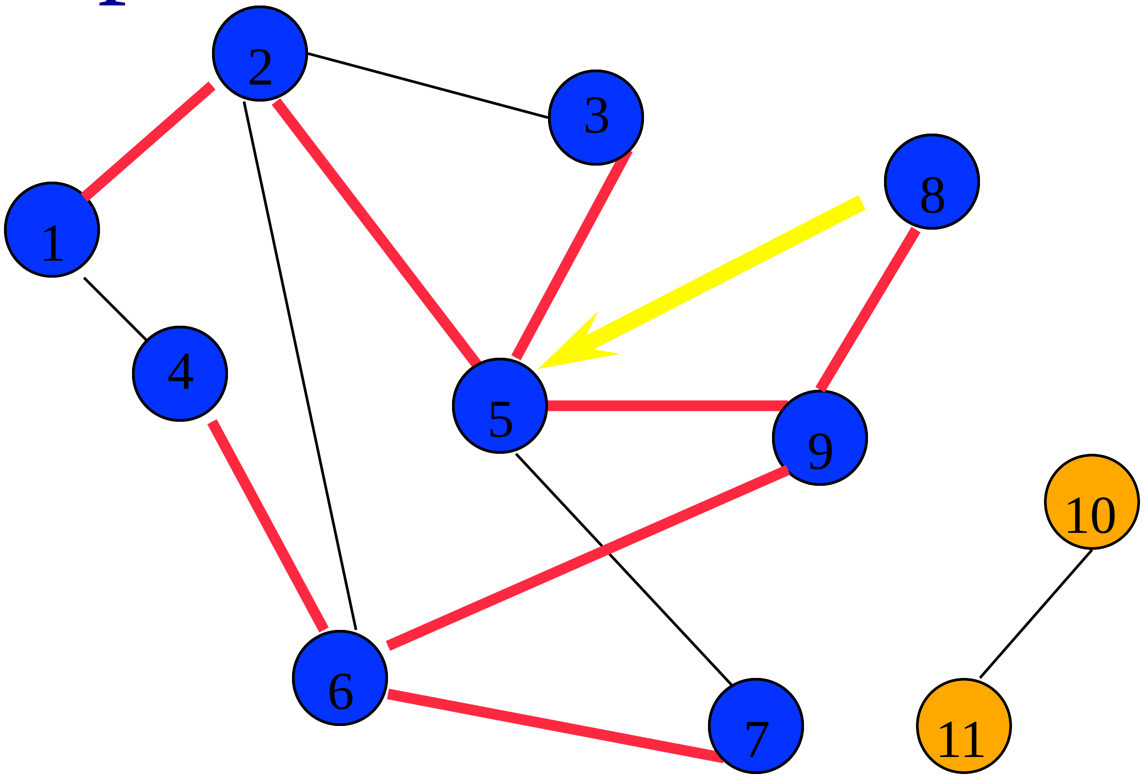
Return to 5.

Depth-First Search Example



Do a dfs(3).

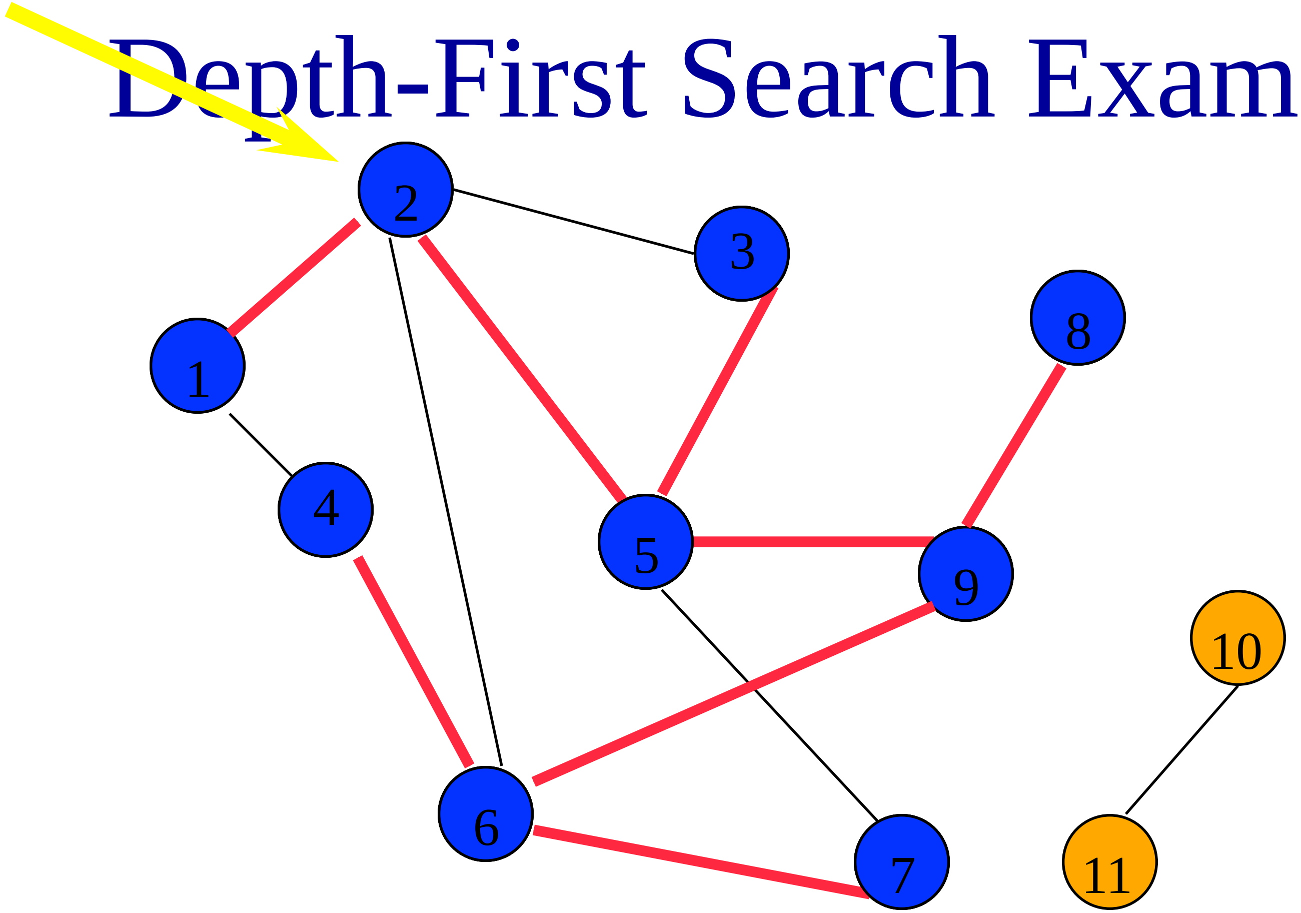
Depth-First Search Example



Label 3 and return to 5.

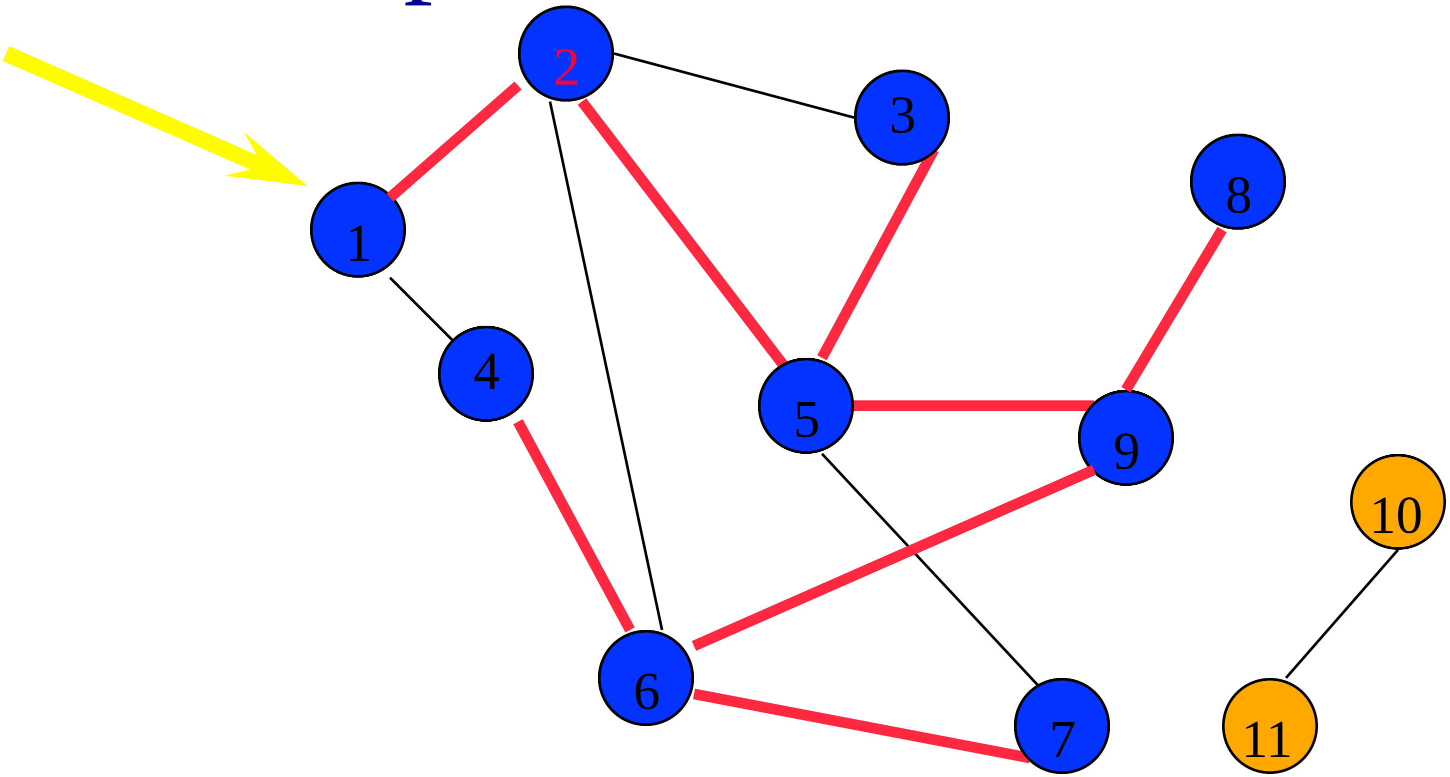
Return to 2.

Depth-First Search Example



Return to 1.

Depth-First Search Example



Return to invoking method.

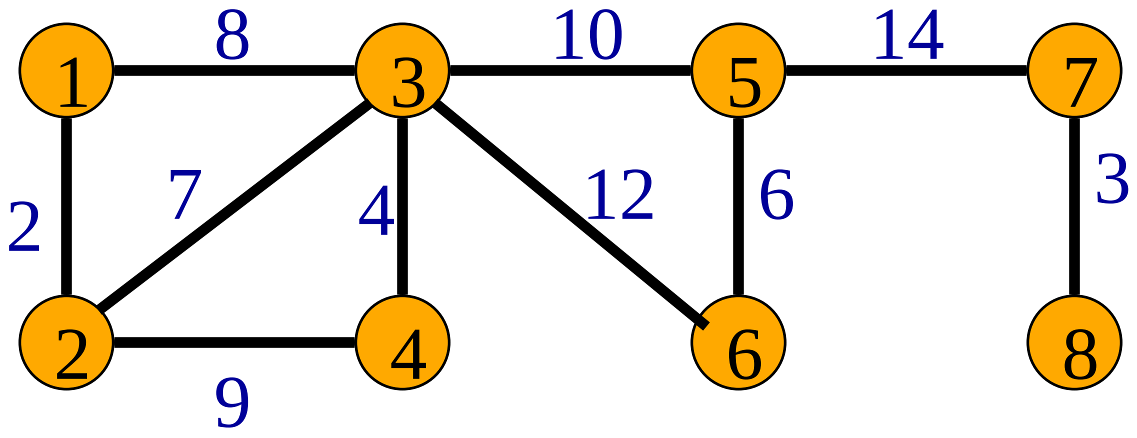
Depth-First Search Properties

- Same complexity as BFS.
- Same properties with respect to path finding, connected components, and spanning trees.
- Edges used to reach unlabeled vertices define a depth-first spanning tree when the graph is connected.
- There are problems for which bfs is better than dfs and vice versa.

Minimum-Cost Spanning Tree

- weighted connected undirected graph
- spanning tree
- cost of spanning tree is sum of edge costs
- find spanning tree that has minimum cost

Example



- Network has 10 edges.
- Spanning tree has only n - 1 = 7 edges.
- Need to either select 7 edges or discard 3.

Greedy Method

- Solve problem by making a sequence of decisions.
- Decisions are made one by one in some order.
- Each decision is made using a greedy criterion.
- A decision, once made, is (usually) not changed later.

Edge Selection Greedy Strategies

- Start with an n-vertex 0-edge forest. Consider edges in ascending order of cost. Select edge if it does not form a cycle together with already selected edges.
  - Kruskal's method.
- Start with a 1-vertex tree and grow it into an n-vertex tree by repeatedly adding a vertex and an edge. When there is a choice, add a least cost edge.
  - Prim's method.

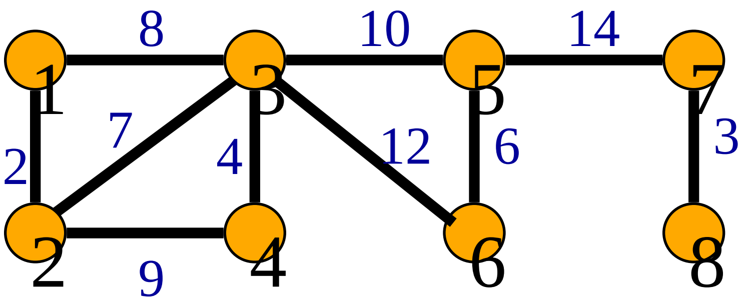
Edge Selection Greedy Strategies

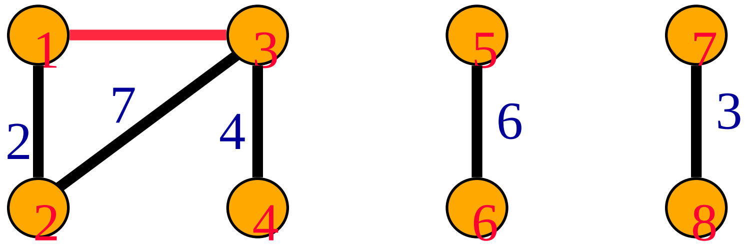
- Start with an n-vertex forest. Each component/tree selects a least cost edge to connect to another component/tree. Eliminate duplicate selections and possible cycles. Repeat until only 1 component/tree is left.
  - Sollin’s method.

Edge Rejection Greedy Strategies

- Start with the connected graph. Repeatedly find a cycle and eliminate the highest cost edge on this cycle. Stop when no cycles remain.
- Consider edges in descending order of cost. Eliminate an edge provided this leaves behind a connected graph.

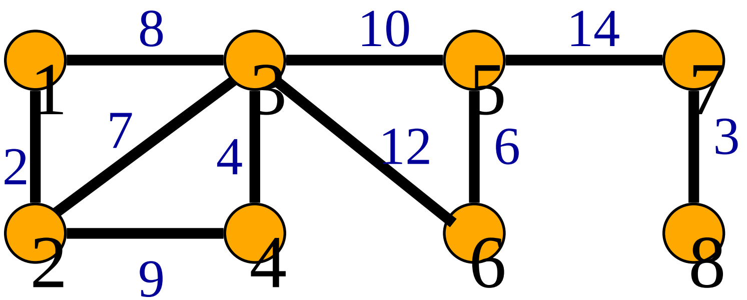
Kruskal’s Method

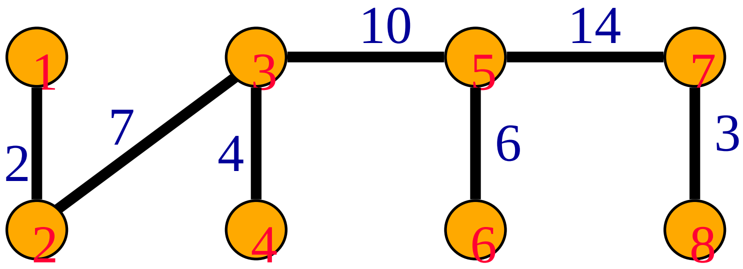




- Edge (7,8) is considered next and added.
- Edge (3,4) is considered next and added.
- Edge (5,6) is considered next and added.
- Edge (2,3) is considered next and added.
- Edge (1,3) is considered next and rejected because it creates a cycle.

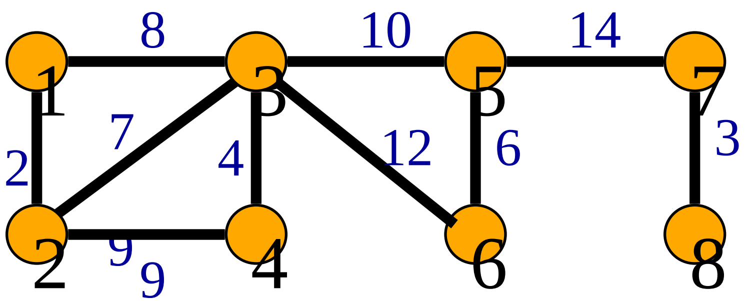
Kruskal’s Method

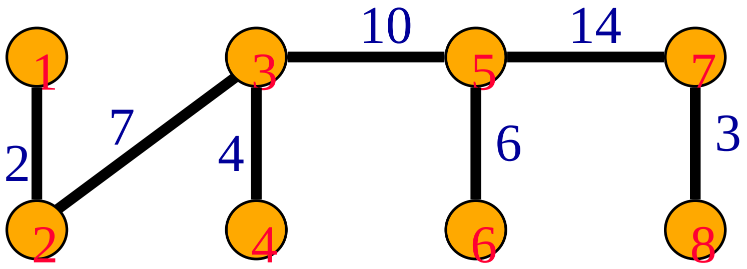




- Edge (2,4) is considered next and rejected because it creates a cycle.
- Edge (3,5) is considered next and added.
- Edge (3,6) is considered next and rejected.
- Edge (5,7) is considered next and added.

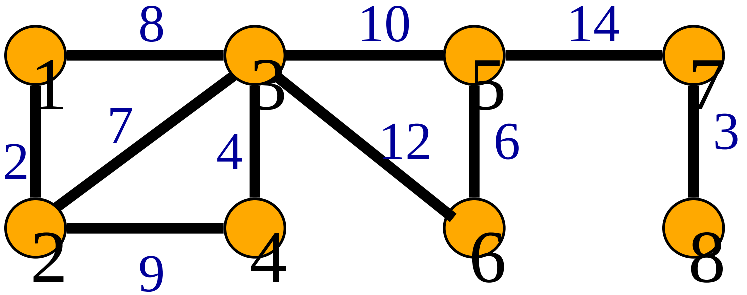
Kruskal’s Method

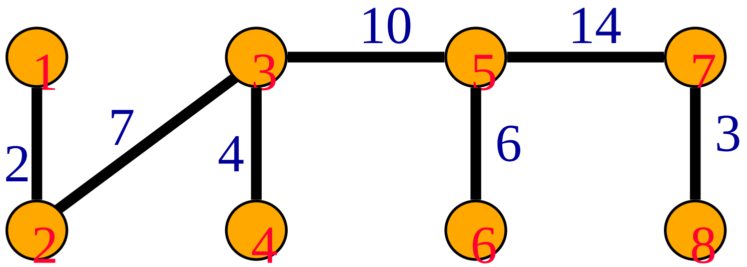




- n - 1 edges have been selected and no cycle formed.
- So we must have a spanning tree.
- Cost is 46.
- Min-cost spanning tree is unique when all edge costs are different.

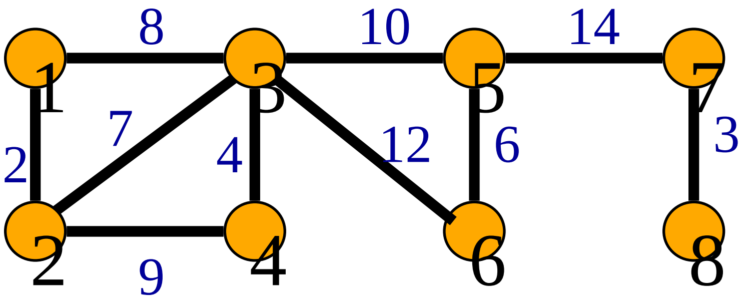
Prim’s Method

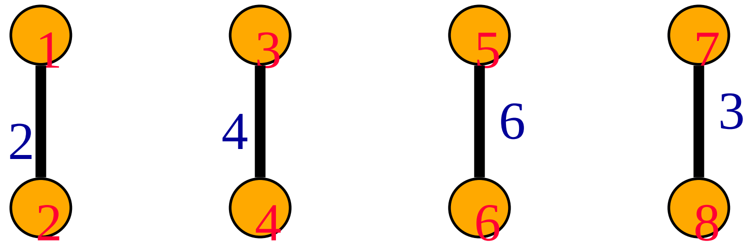




- Start with any single vertex tree.
- Get a 2-vertex tree by adding a cheapest edge.
- Get a 3-vertex tree by adding a cheapest edge.
- Grow the tree one edge at a time until the tree has n - 1 edges (and hence has all n vertices).

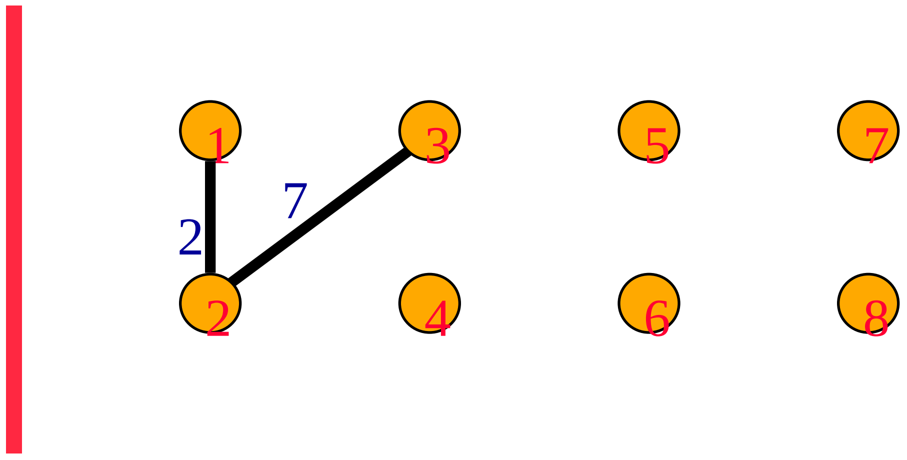
Sollin’s Method





- Start with a forest that has no edges.
- Each component selects a least cost edge with which to connect to another component.
- Duplicate selections are eliminated.
- Cycles are possible when the graph has

Sollin's Method



- Each component that remains selects a least cost edge with which to connect to another component.
- Beware of duplicate selections and cycles.

Greedy Minimum-Cost Spanning Tree Methods

- Can prove that all result in a minimum-cost spanning tree.
- See Text Book

Pseudocode For Kruskal’s Method

```
Start with an empty set T of edges.
while (E is not empty && |T| != n-1)
{
    Let (u,v) be a least-cost edge in E.
    E = E - {(u,v)}. // delete edge from E
    if ((u,v) does not create a cycle in T)
       Add edge (u,v) to T.
}
if (| T | == n-1) T is a min-cost spanning tree.
else Network has no spanning tree.
```

Data Structures For Kruskal’s Method

Edge set E.

Operations are:

- Is E empty?
- Select and remove a least-cost edge.

Use a min heap of edges.

- Initialize. O(e) time.
- Remove and return least-cost edge. O(log e) time.

Data Structures For Kruskal's Method

Set of selected edges T.

Operations are:

- Does T have n - 1 edges?
- Does the addition of an edge (u, v) to T result in a cycle?
- Add an edge to T.

Data Structures For Kruskal’s Method

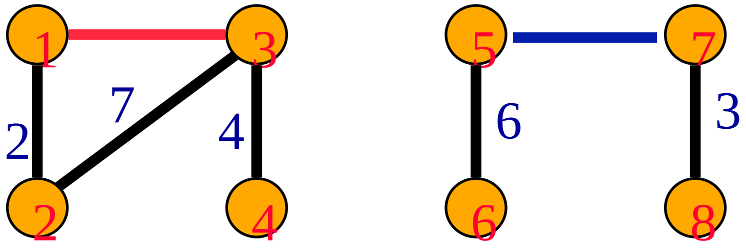
Use an array linear list for the edges of T.

- Does T have n - 1 edges?
  - Check size of linear list. O(1) time.
- Does the addition of an edge (u, v) to T result in a cycle?
  - Not easy.
- Add an edge to T.
  - Add at right end of linear list. O(1) time.

Just use an array rather than ArrayLinearList.

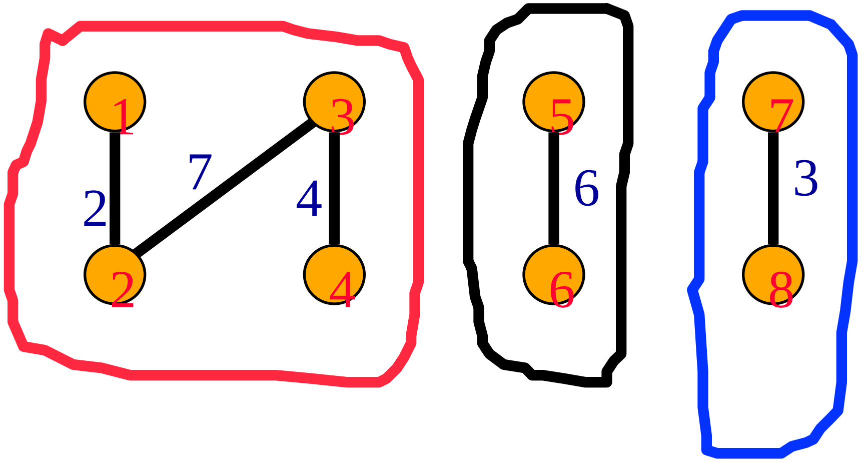
Data Structures For Kruskal's Method

Does the addition of an edge (u, v) to T result in a cycle?



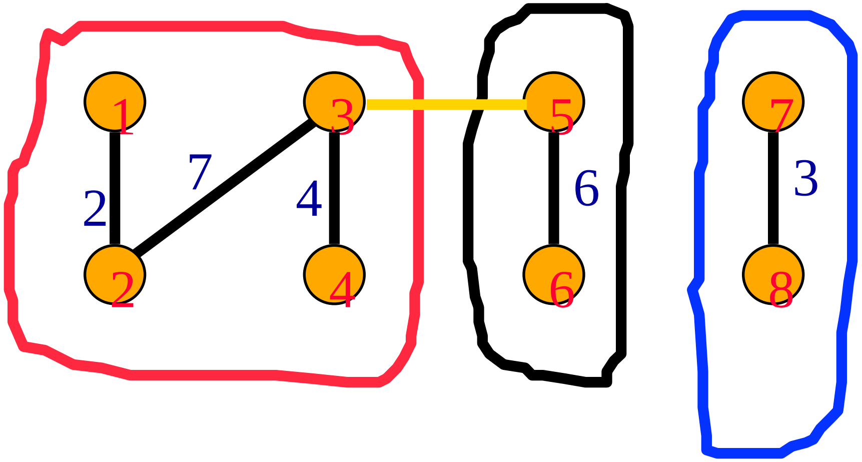
- Each component of T is a tree.
- When u and v are in the same component, the addition of the edge (u,v) creates a cycle.
- When u and v are in the different components, the addition of the edge (u,v) does not create a cycle.

Data Structures For Kruskal's Method



- Each component of T is defined by the vertices in the component.
- Represent each component as a set of vertices.
  - {1, 2, 3, 4}, {5, 6}, {7, 8}
- Two vertices are in the same component iff they are in the same set of vertices.

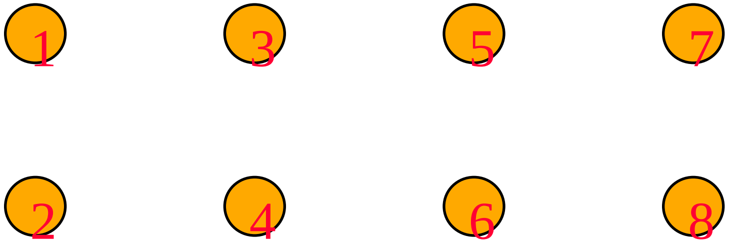
Data Structures For Kruskal's Method



- When an edge (u, v) is added to T, the two components that have vertices u and v combine to become a single component.
- In our set representation of components, the set that has vertex u and the set that has vertex v are united.
    - {1, 2, 3, 4} + {5, 6} => {1, 2, 3, 4, 5, 6}

Data Structures For Kruskal's Method

- Initially, T is empty.



- Initial sets are:
  - {1} {2} {3} {4} {5} {6} {7} {8}
- Does the addition of an edge (u, v) to T result in a cycle? If not, add edge to T.

```
s1 = find(u); s2 = find(v);

if
```

Data Structures For Kruskal's Method

- Use FastUnionFind.
- Initialize.
  - O(n) time.
- At most 2e finds and n-1 unions.
  - Very close to O(n + e).
- Min heap operations to get edges in increasing order of cost take O(e log e).
- Overall complexity of Kruskal's method is O(n + e log e).

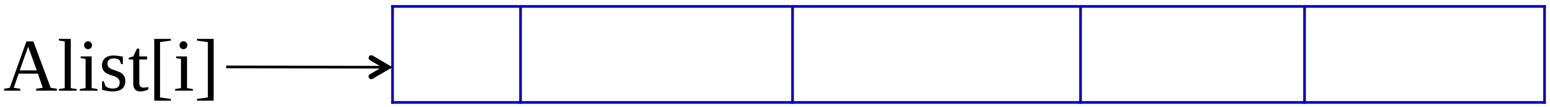
Greedy Minimum-Cost Spanning Tree Methods

- Prim’s method is fastest.
  - $O(n^2)$ using an implementation similar to that of Dijkstra’s shortest-path algorithm.
  - $O(e + n \log n)$ using a Fibonacci heap.
- Kruskal’s uses union-find trees to run in $O(n + e \log e)$ time.

- Implement a full version algorithm of Kruskal’s Method (**Experiment**)

- Implement a BFS algorithm using Adjacency Multilists

Adjacency Multilists



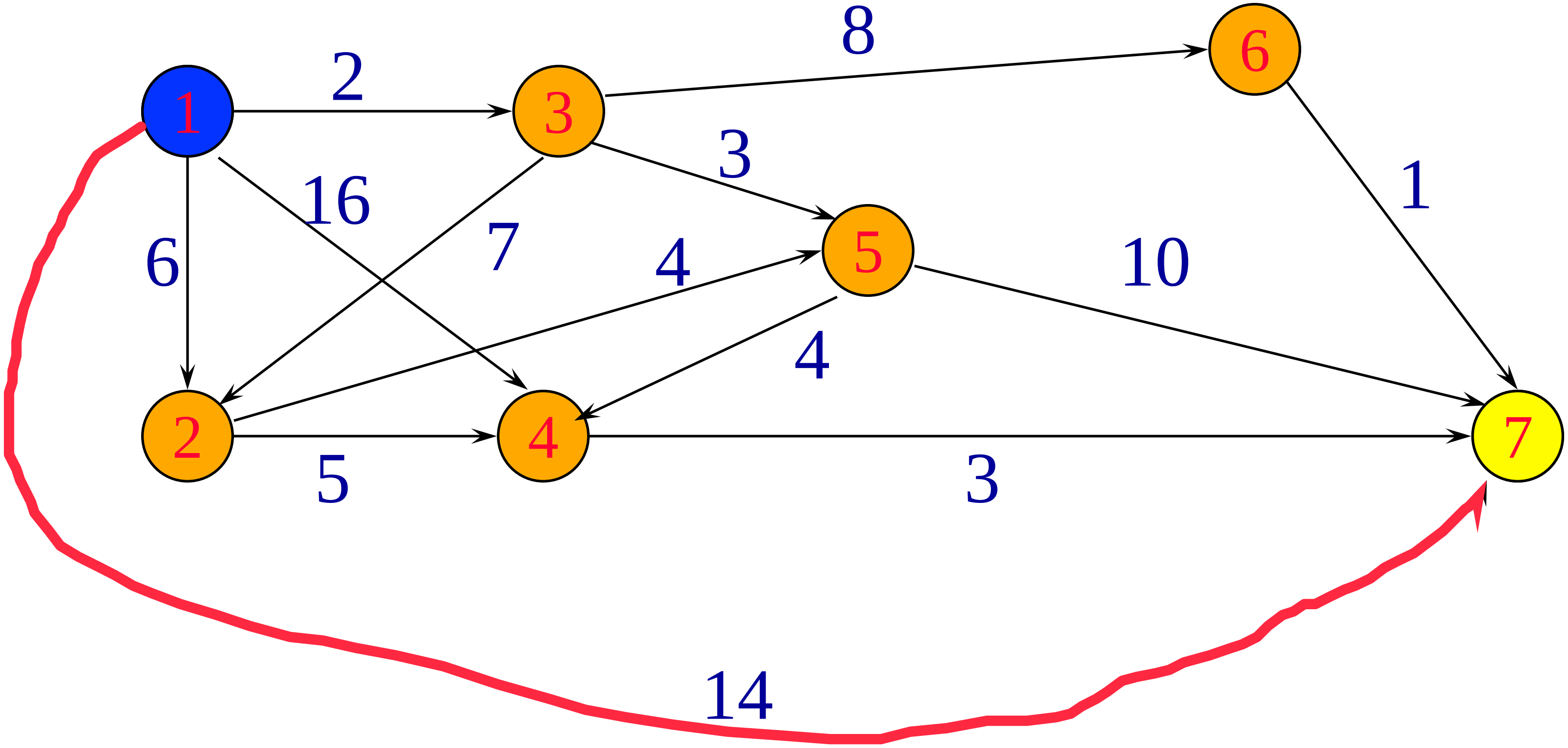
- **virtual void** Graph::BFS (**int** v) **{**
- visited = **new bool**[n]; fill(visited, visited + n, **false**);
- visited[v] = **true;**
- Queue<**int**> q**;**
- q.Push(v)**;**
- **while** (!q.IsEmpty()) **{**
- v = q.Front(); q.Pop();
- **ADNode * p = Alist[v];**
- while(p != null){

```
     int w ;
     if(p->v1 == v) {
         w = p->v2;
         p = p->v1link;}
     else{
         w = p->v1;
         p = p->v2link;}
     if (!visited[w]) {
        q.Push(w);
        visited[w] = true;
    }  // end of while(p)
     }          end of while(q)
```

Shortest Path Problems

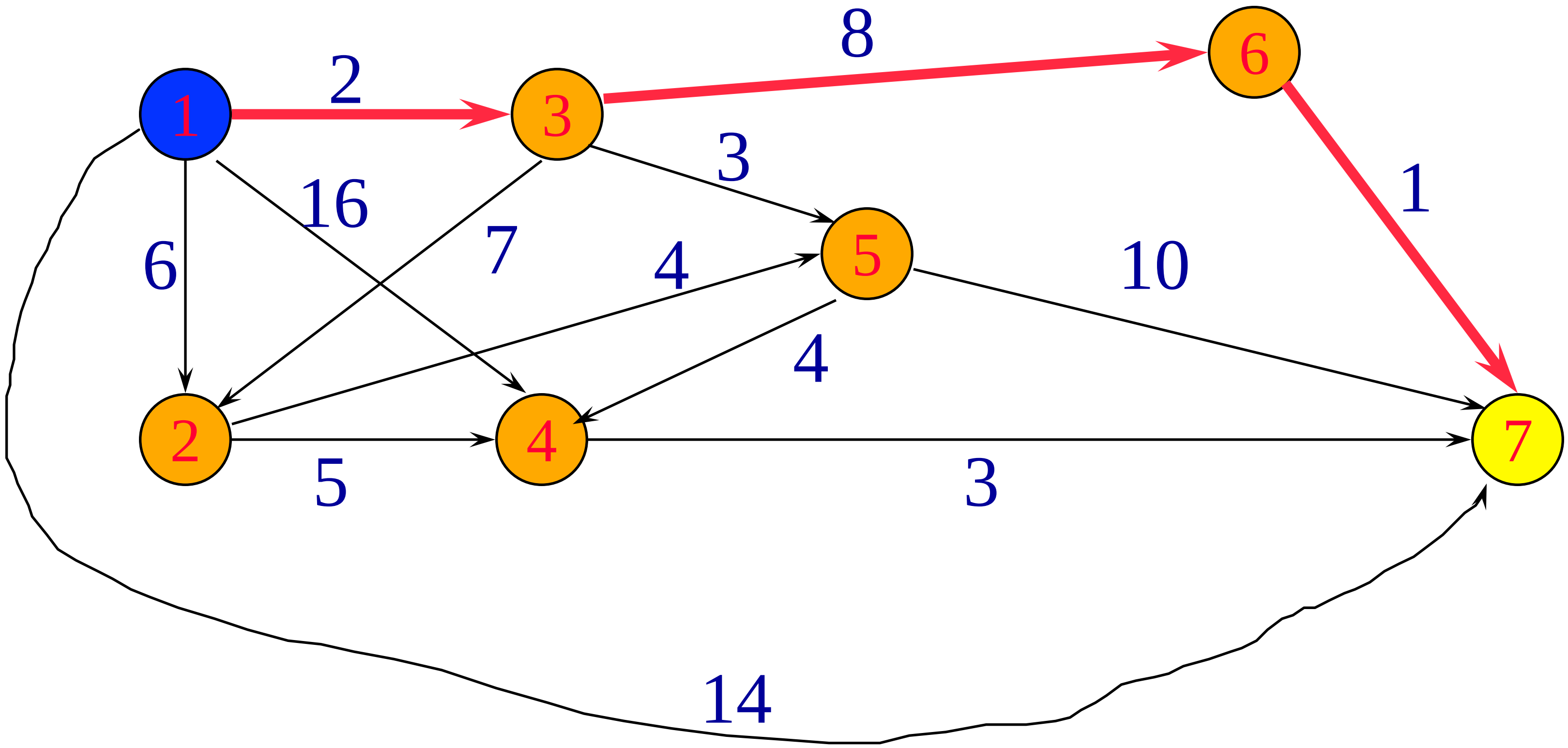
- Directed weighted graph.
- Path length is sum of weights of edges on path.
- The vertex at which the path begins is the source vertex.
- The vertex at which the path ends is the destination vertex.

Example



A path from 1 to 7.
Path length is 14.

Example



Another path from 1 to 7.

Path length is 11.

Shortest Path Problems

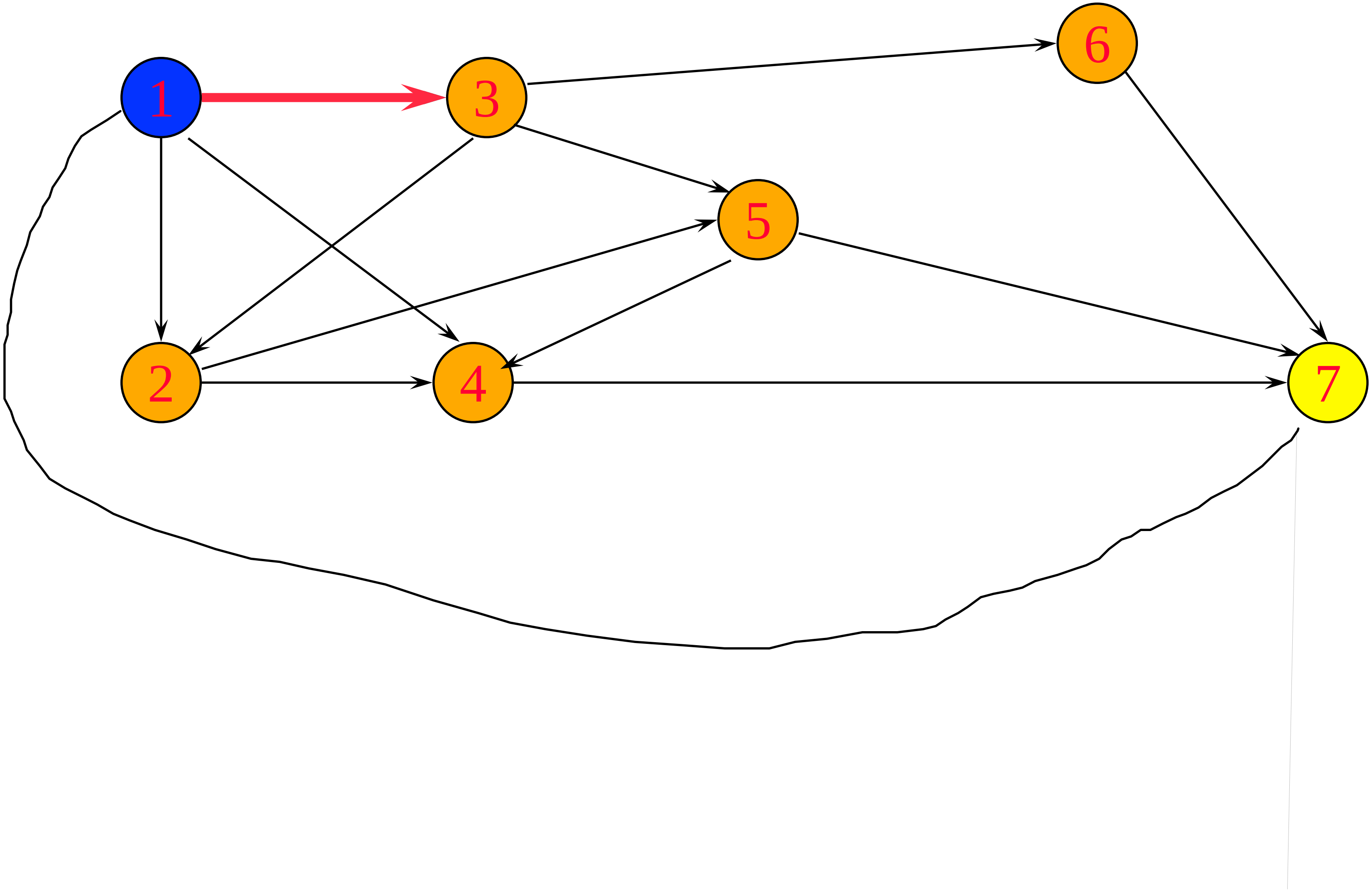
- Single source single destination.
- Single source all destinations.
- All pairs (every vertex is a source and destination).

Single Source Single Destination

Possible greedy algorithm:

- Leave source vertex using cheapest/shortest edge.
- Leave new vertex using cheapest edge subject to the constraint that a new vertex is reached.
- Continue until destination is reached.

Greedy Shortest 1 To 7 Path



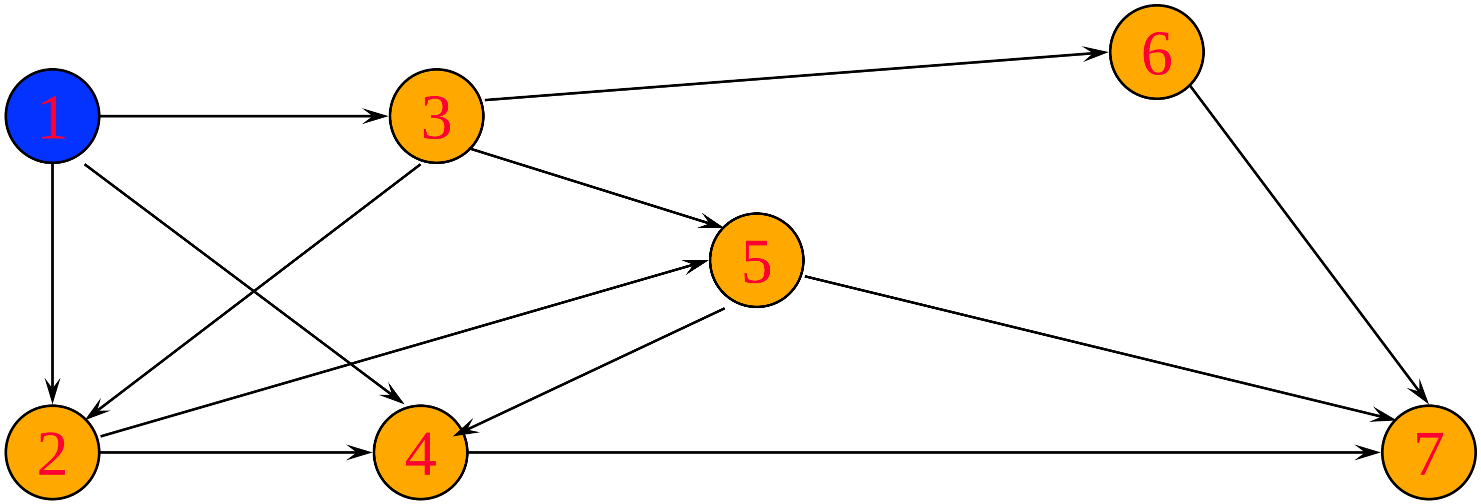
Single Source All Destinations

Need to generate up to n (n is number of vertices) paths (including path from source to itself).

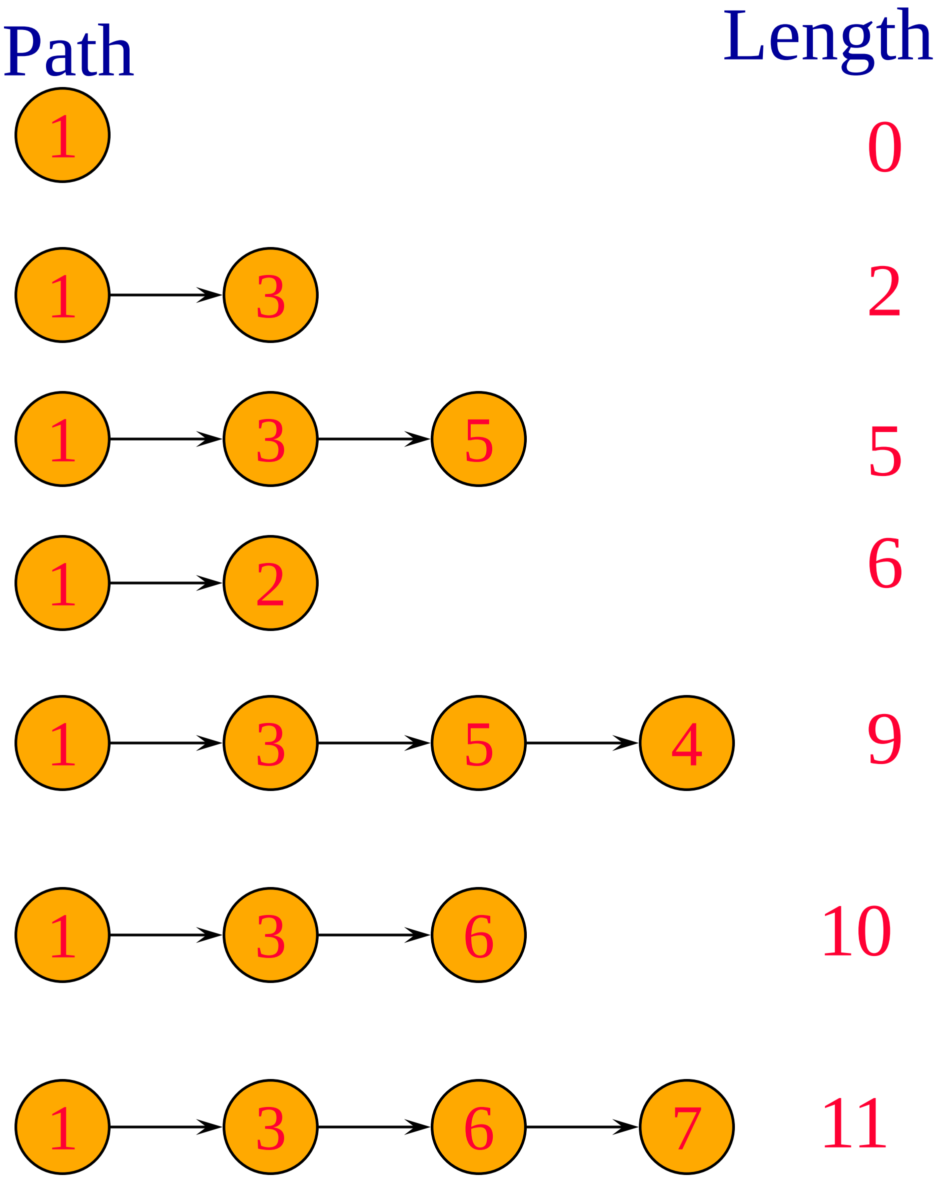
Greedy method:

- Construct these up to n paths in order of increasing length.
- Assume edge costs (lengths) are >= 0.
- So, no path has length < 0.
- First shortest path is from the source vertex to itself. The length of this path is 0.

Greedy Single Source All Destinations



Greedy Single Source All Destinations



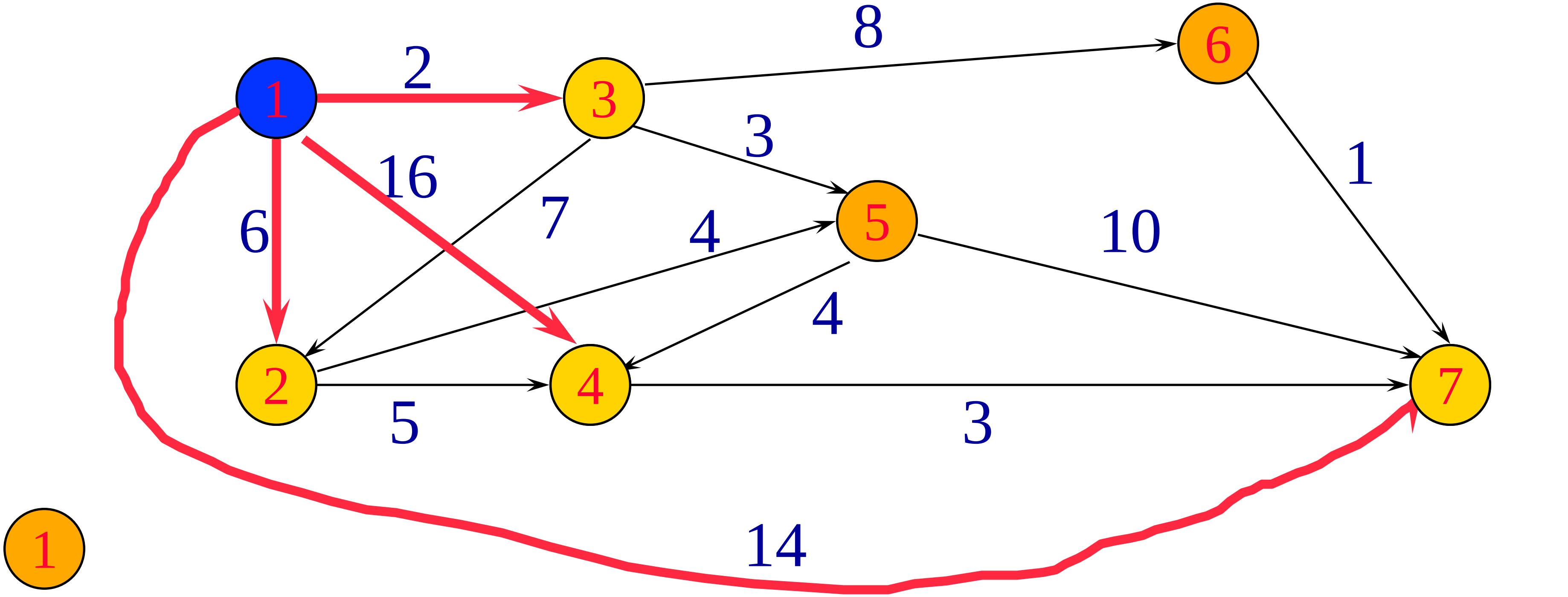
- Each path (other than first) is a one edge extension of a previous path.

- Next shortest path is the shortest one edge extension of an already generated shortest path.

Greedy Single Source All Destinations

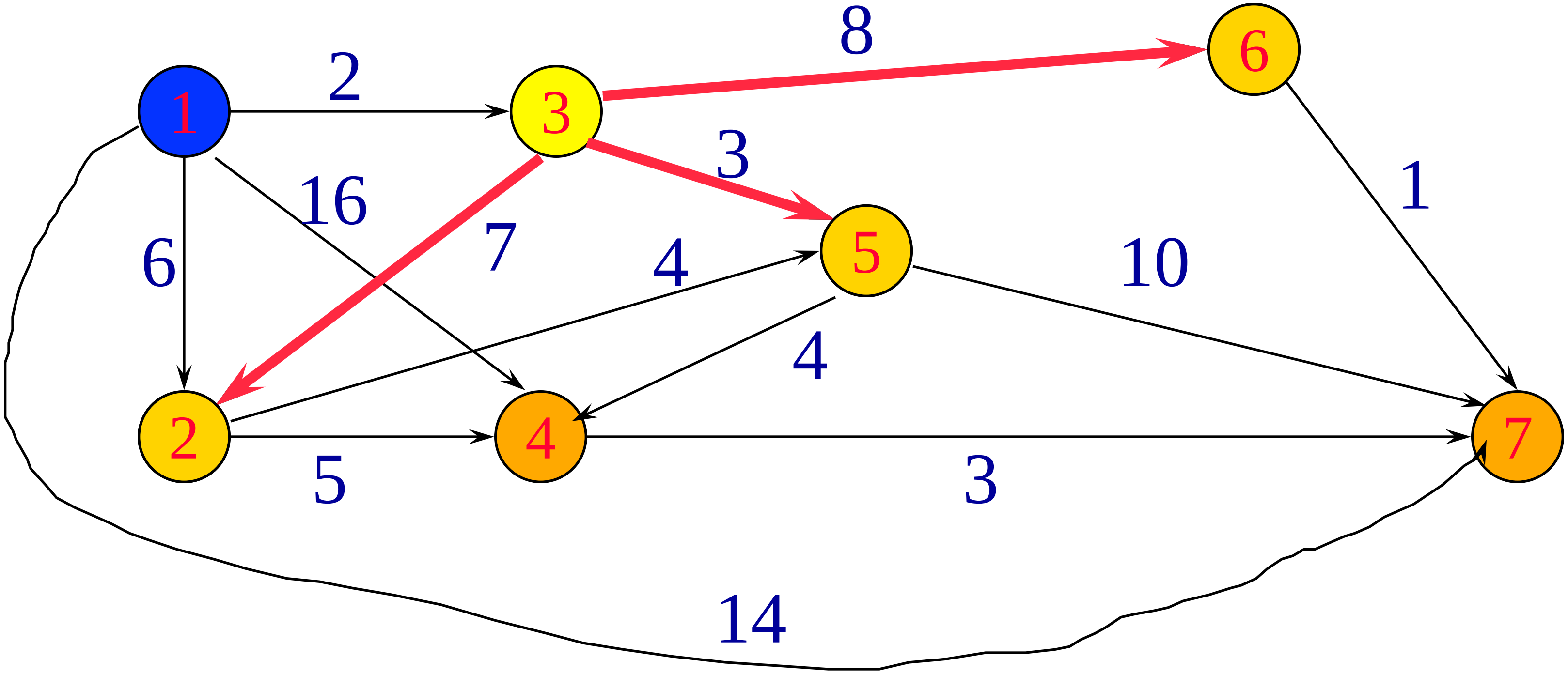
- Let d(i) (distanceFromSource(i)) be the length of a shortest one edge extension of an already generated shortest path, the one edge extension ends at vertex i.
- The next shortest path is to an as yet unreached vertex for which the d() value is least.
- Let p(i) (predecessor(i)) be the vertex just before vertex i on the shortest one edge extension to i.

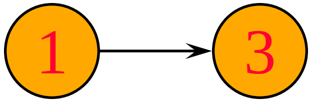
Greedy Single Source All Destinations



	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	16	-	-	14
p	-	1	1	1	-	-	1

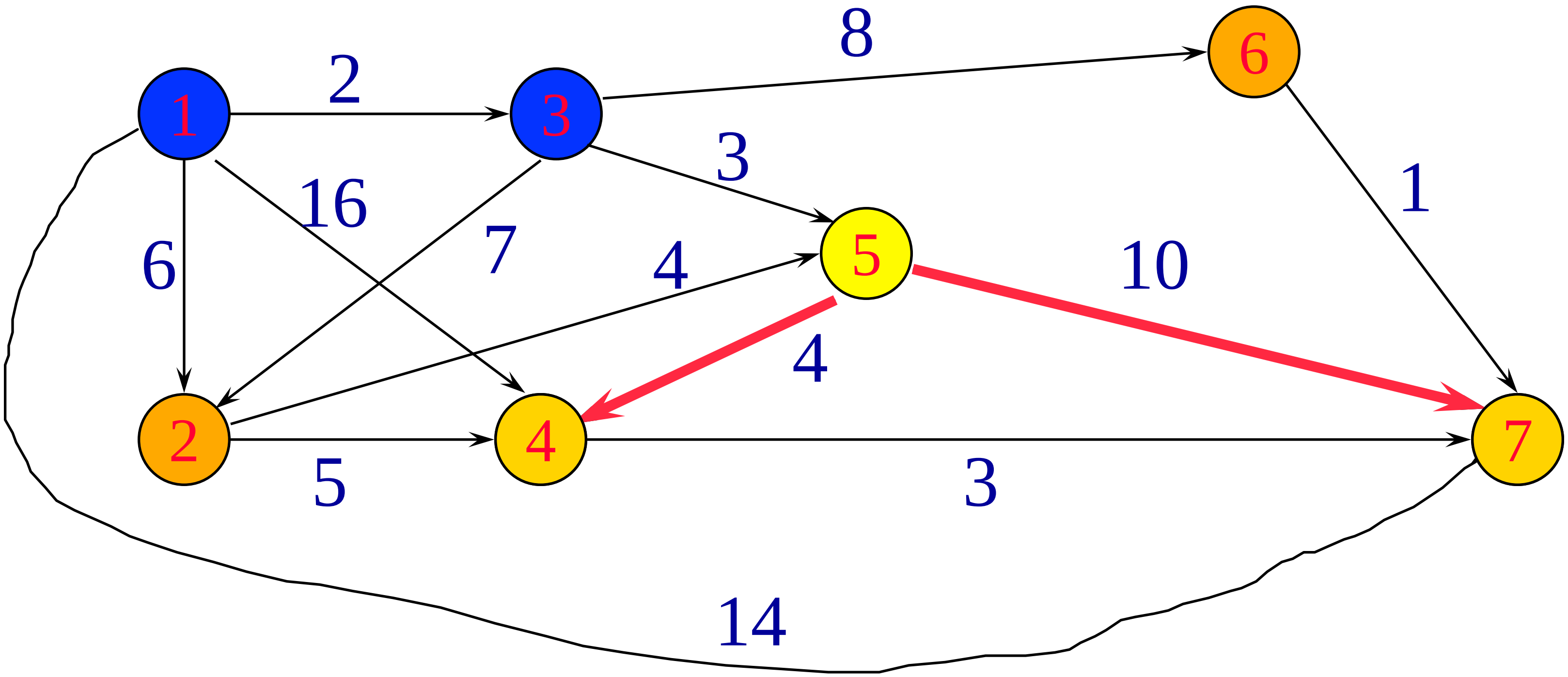
Greedy Single Source All Destinations

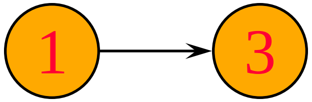


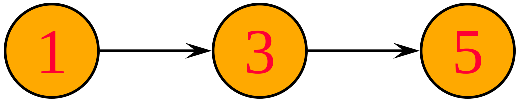


	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	16	5	10	14
p	-	1	1	1	3	3	1

Greedy Single Source All Destinations

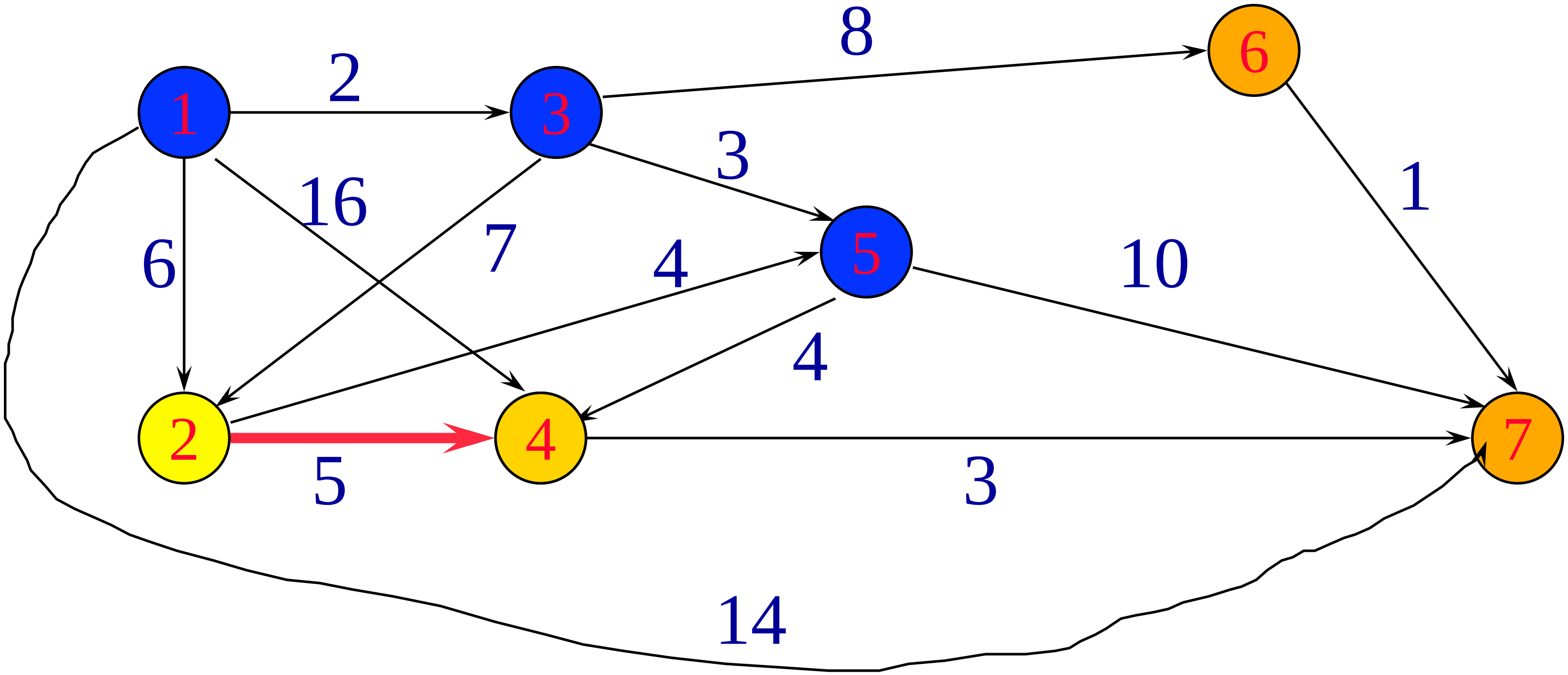


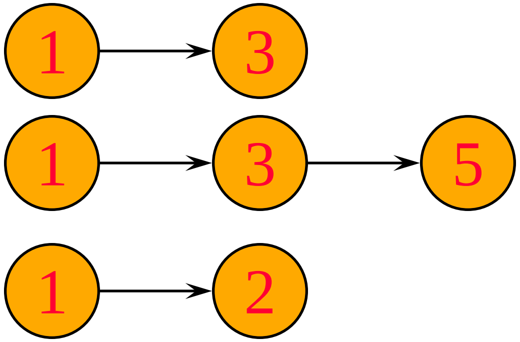




	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	9	5	10	14
p	-	1	1	5	3	3	1

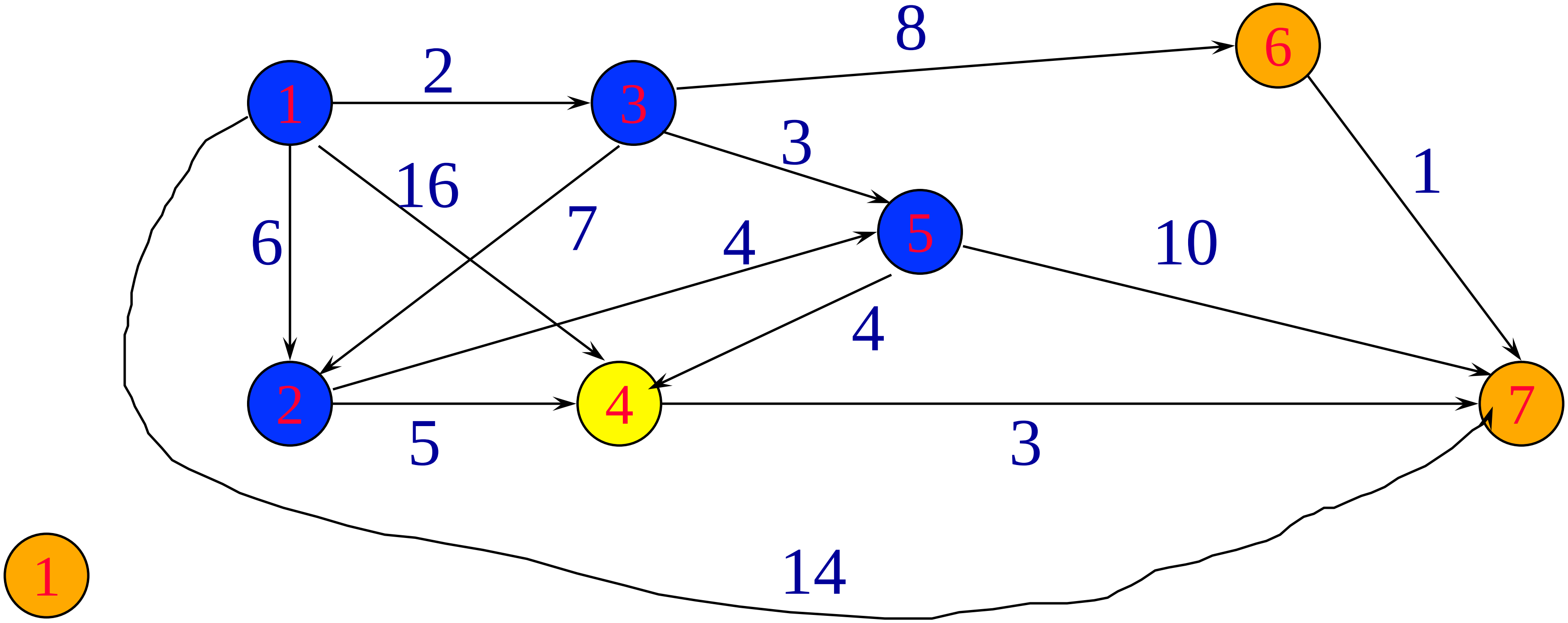
Greedy Single Source All Destinations





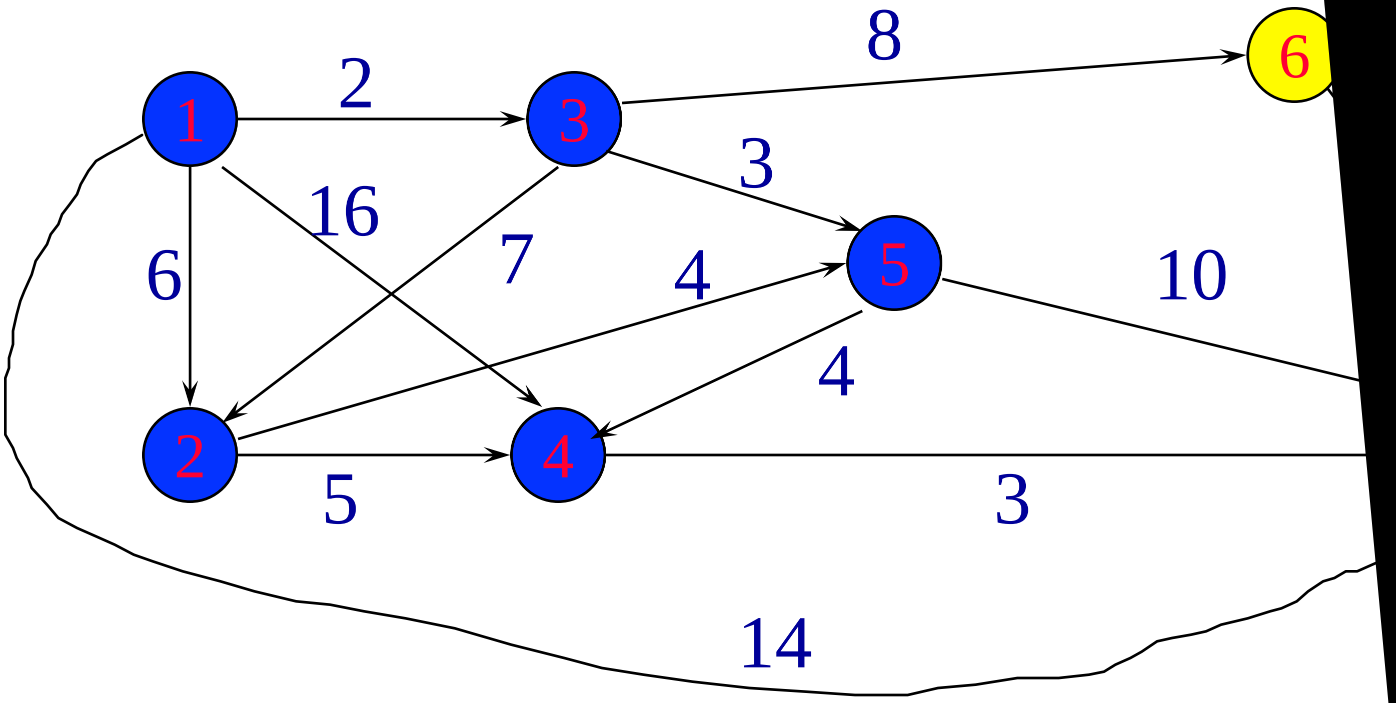
	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	9	5	10	14
p	-	1	1	5	3	3	1

Greedy Single Source All Destinations



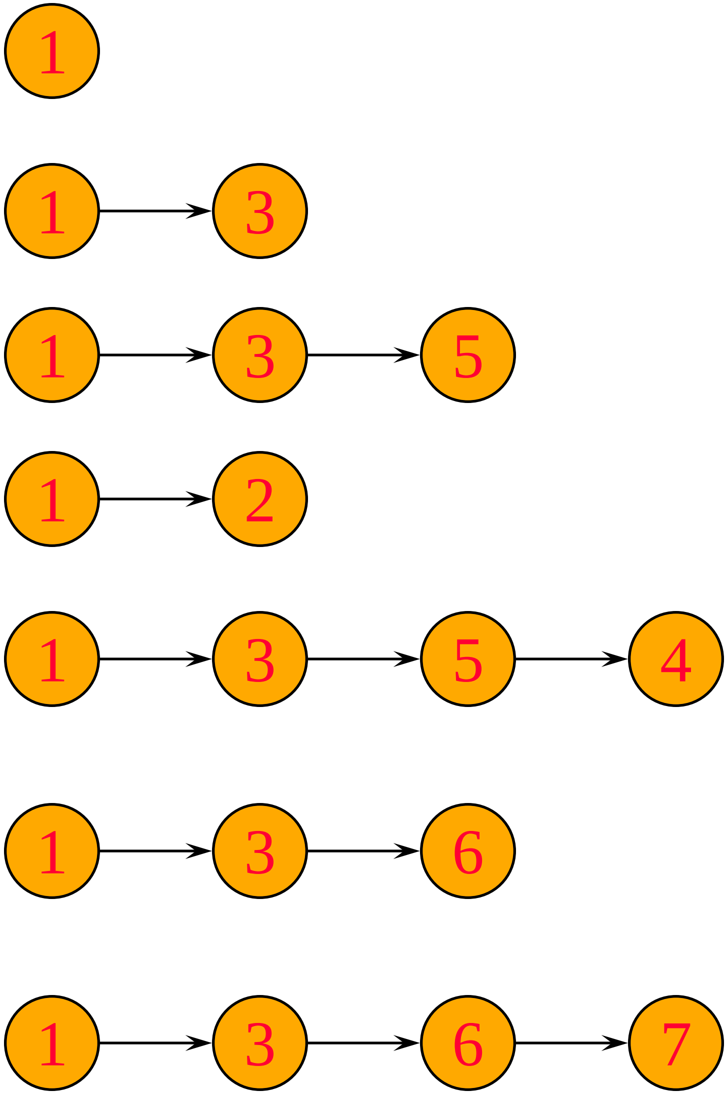
	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	9	-	10	14
p	-	1	1	5	-	-	1

Greedy Single Source All Desti [redacted]



	[1]	[2]	[3]	[4]	[5]	[redacted]	[redacted]
d	0	6	2	9	5	[redacted]	[redacted]
p	-	1	1	5		-	1

Greedy Single Source All Destinations



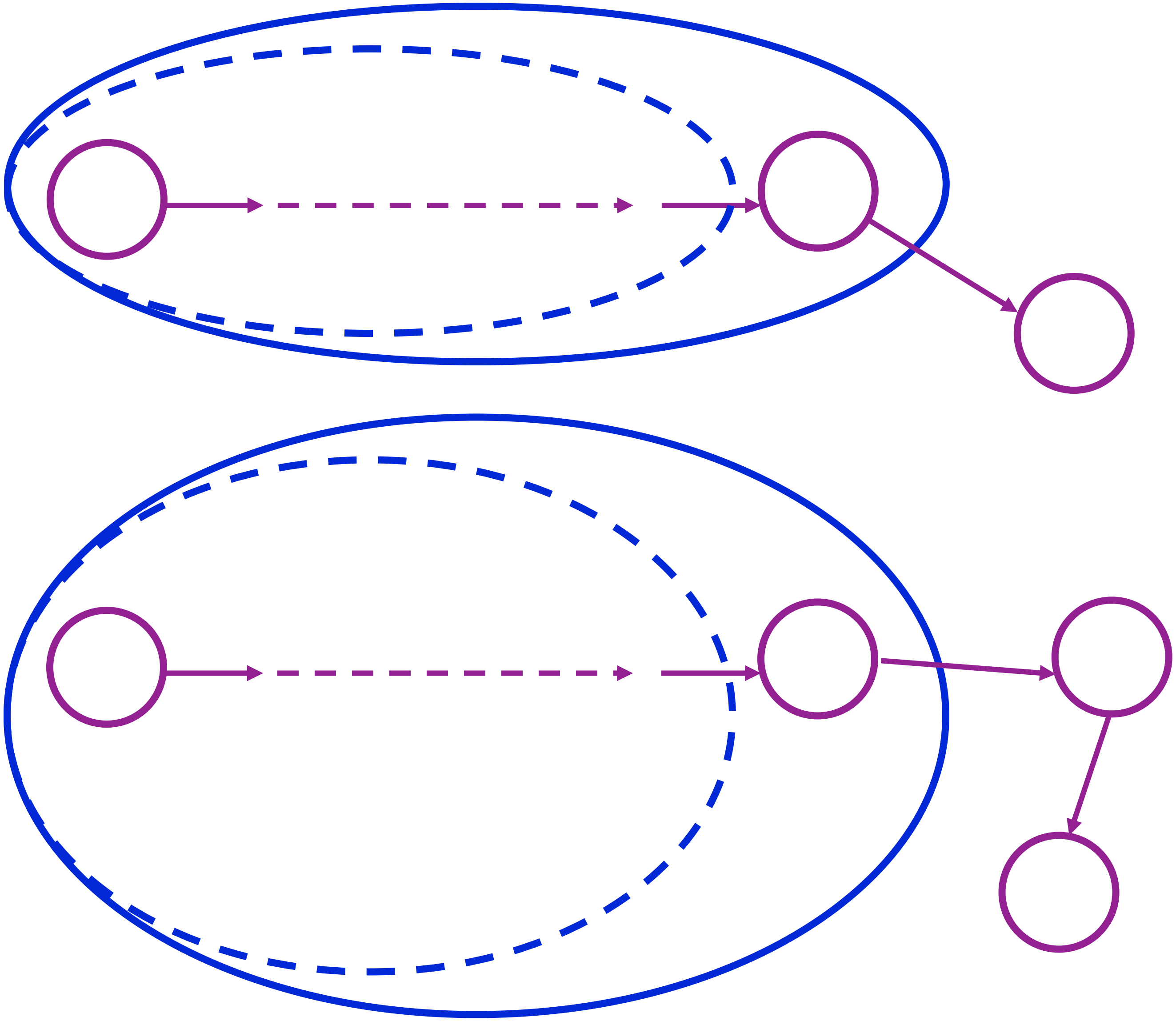
Path	Length
1	0
1 → 3	2
1 → 3 → 5	5
1 → 2	6
1 → 3 → 5 → 4	9
1 → 3 → 6	10
1 → 3 → 6 → 7	11

[1]	[2]	[3]	[4]	[5]	[6]	[7]
0	6	2	9	5	10	11
-	1	1	5	3	3	6

Single Source Single Destination

Terminate single source all destinations greedy algorithm as soon as shortest path to desired vertex has been generated.

Correctness



Data Structures For Dijkstra’s Algorithm

- The greedy single source all destinations algorithm is known as Dijkstra’s algorithm.
- Implement d() and p() as 1D arrays.
- Keep a linear list L of reachable vertices to which shortest path is yet to be generated.
- Select and remove vertex v in L that has smallest d() value.
- Update d() and p() values of vertices adjacent to v.

```
void MatrixDigraph::ShortestPath( int n, int v){
    for (int i=0; i<n; i++) {
            L[i]=false; dist[i]=length[v][i];}
    L[v]=true;
    dist[v]=0;
    for (i=0; i<n-2; i++) { //determine n-1 paths from v
```

Complexity

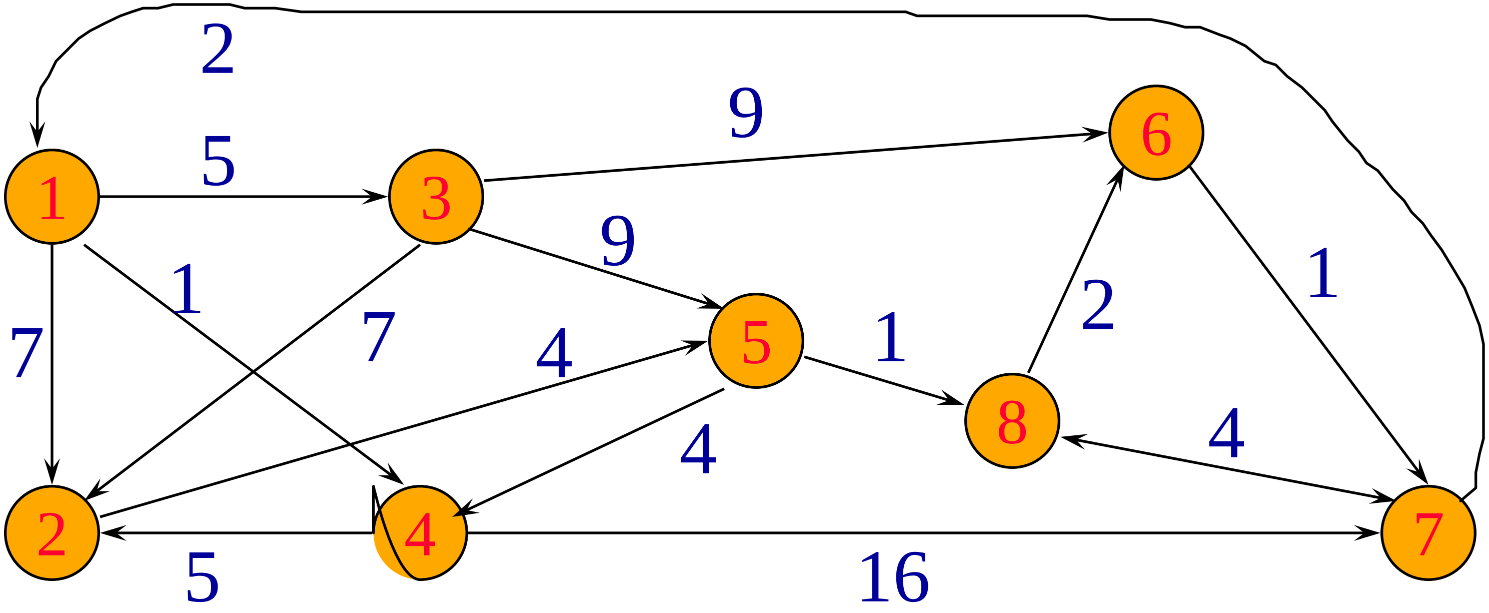
- O(n) to select next destination vertex.
- O(out-degree) to update d() and p() values when adjacency lists are used.
- O(n) to update d() and p() values when adjacency matrix is used.
- Selection and update done once for each vertex to which a shortest path is found.
- Total time is $O(n^2 + e) = O(n^2)$.

Complexity

- When a min heap of d() values is used in place of the linear list L of reachable vertices, total time is O((n+e) log n), because O(n) remove min operations and O(e) change key (d() value) operations are done.
- When e is $O(n^2)$, using a min heap is worse than using a linear list.
- When a Fibonacci heap is used, the total time is O(n log n + e).

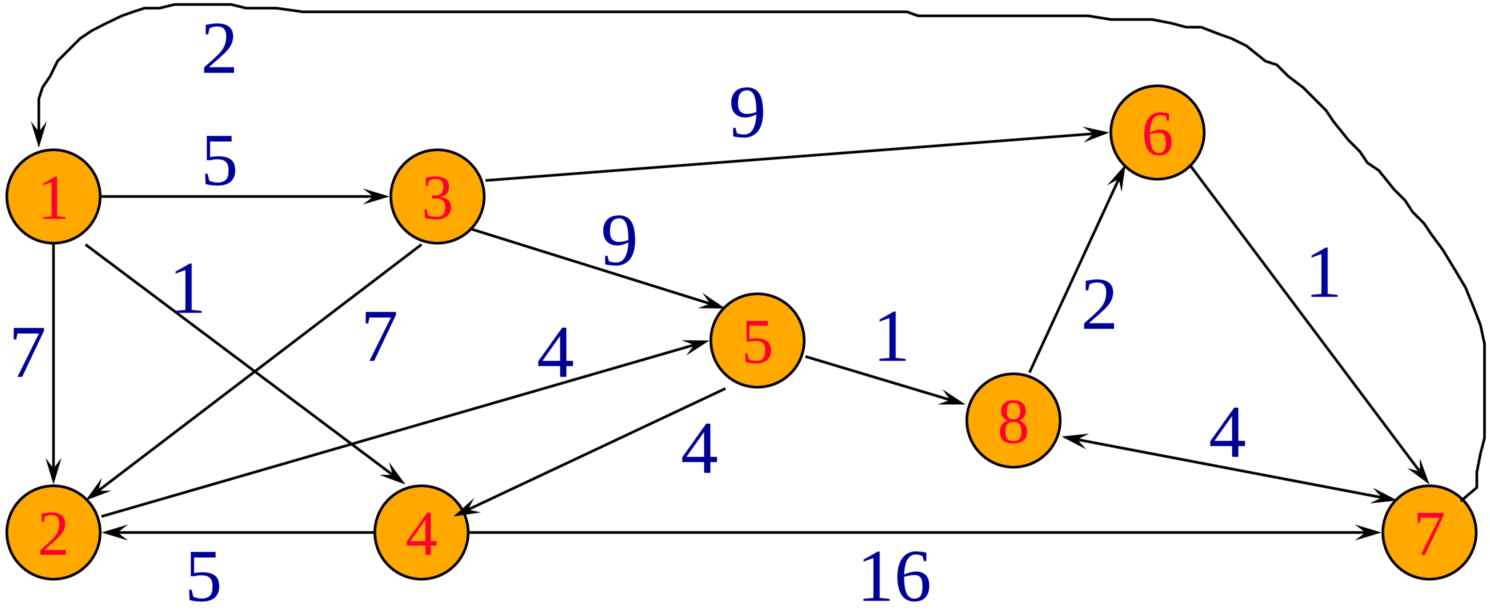
All-Pairs Shortest Paths

- Given an n-vertex directed weighted graph, find a shortest path from vertex i to vertex j for each of the n^2 vertex pairs (i,j).

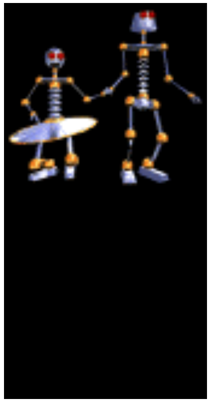


Dijkstra’s Single Source Algorithm

- Use Dijkstra’s algorithm n times, once with each of the n vertices as the source vertex.



Performance

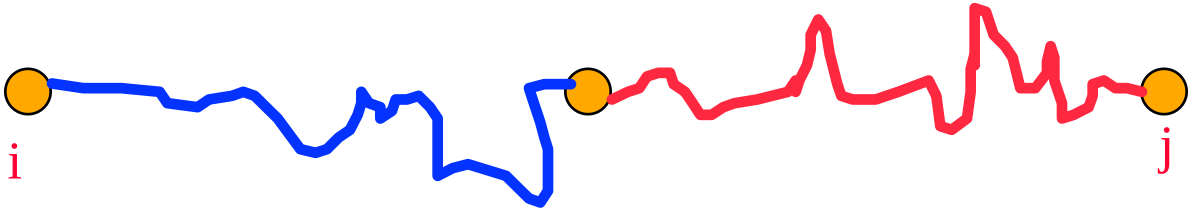


- Time complexity is $O(n^3)$ time.
- Works only when no edge has a cost < 0.

Dynamic Programming Solution

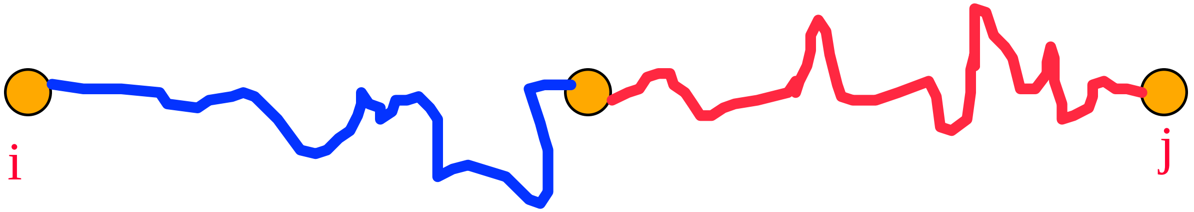
- Time complexity is Theta(n^3) time.
- Works so long as there is no cycle whose length is < 0.
- When there is a cycle whose length is < 0, some shortest paths aren't finite.
    - If vertex 1 is on a cycle whose length is -2, each time you go around this cycle once you get a 1 to 1 path that is 2 units shorter than the previous one.
- Simpler to code, smaller overheads.
- Known as Floyd's shortest paths algorithm.

Decision Sequence



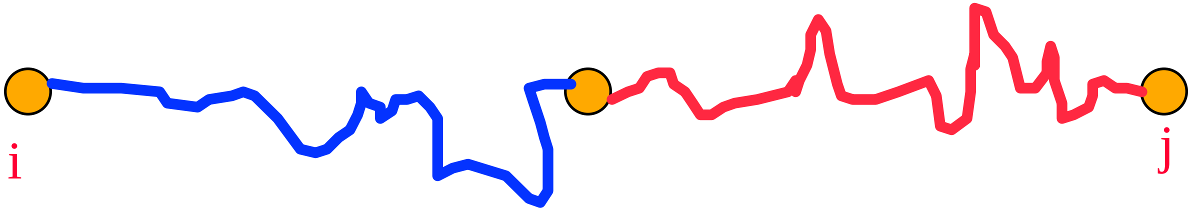
- First decide the highest intermediate vertex (i.e., largest vertex number) on the shortest path from i to j.
- If the shortest path is i, 2, 6, 3, 8, 5, 7, j, the first decision is that vertex 8 is an intermediate vertex on the shortest path and no intermediate vertex is larger than 8.
- Then decide the highest intermediate vertex on the path from i to 8, and so on.

Problem State



- (i,j,k) denotes the problem of finding the shortest path from vertex i to vertex j that has no intermediate vertex larger than k.
- (i,j,n) denotes the problem of finding the shortest path from vertex i to vertex j (with no restrictions on intermediate vertices).

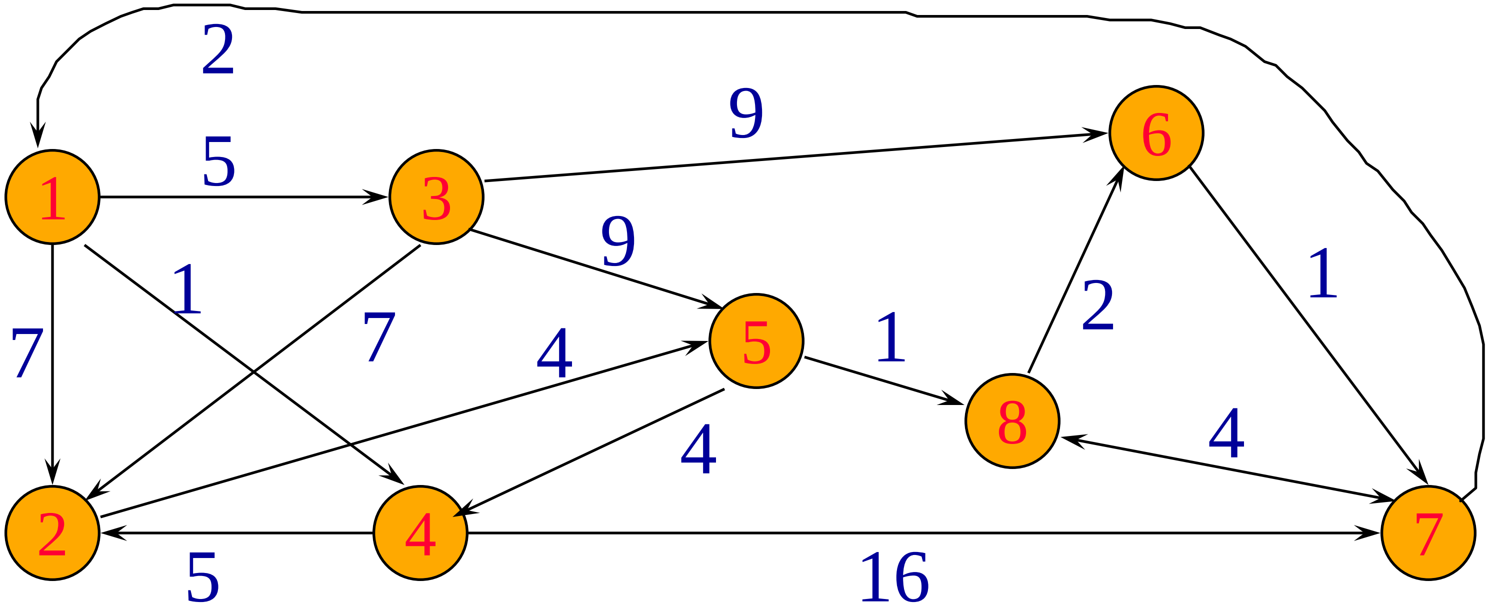
Cost Function



- Let c(i,j,k) be the length of a shortest path from vertex i to vertex j that has no intermediate vertex larger than k.

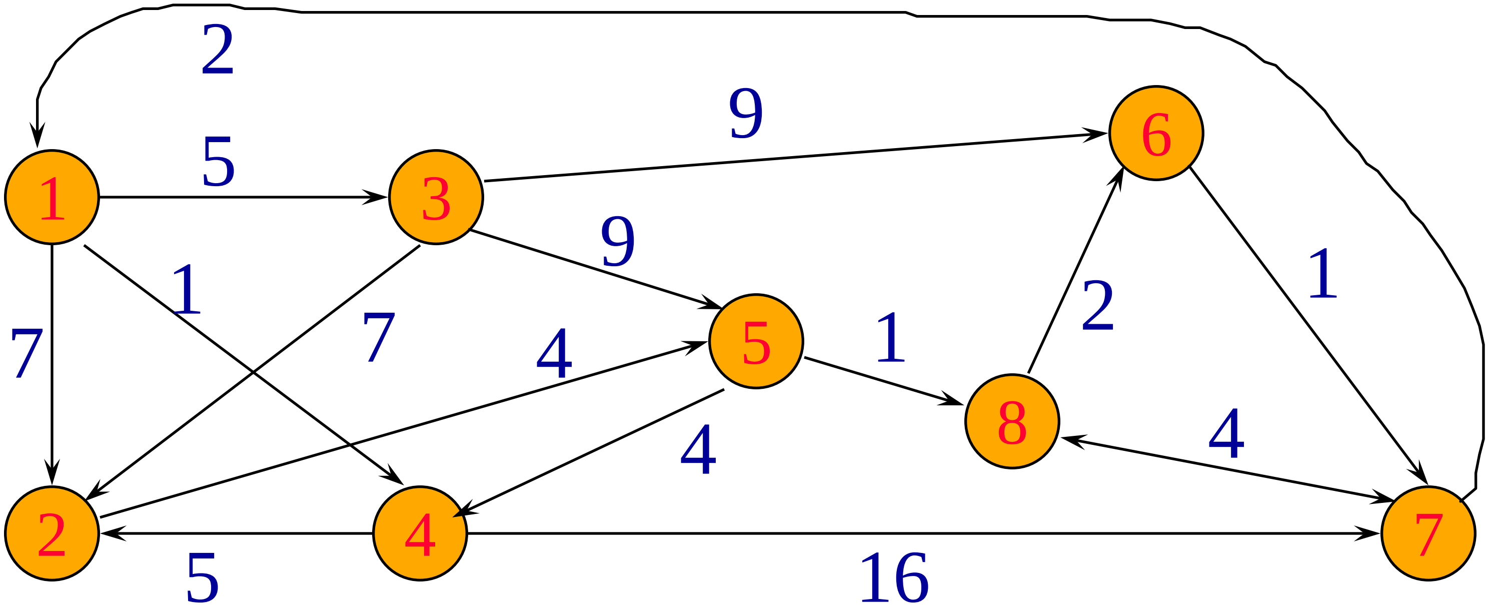
c(i,j,n)

- c(i,j,n) is the length of a shortest path from vertex i to vertex j that has no intermediate vertex larger than n.
- No vertex is larger than n.
- Therefore, c(i,j,n) is the length of a shortest path from vertex i to vertex j.



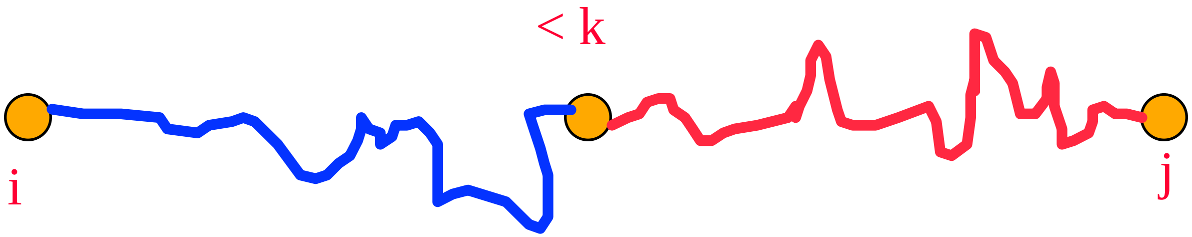
c(i,j,0)

- c(i,j,0) is the length of a shortest path from vertex i to vertex j that has no intermediate vertex larger than 0.
  - Every vertex is larger than 0.
  - Therefore, c(i,j,0) is the length of a single-edge path from vertex i to vertex j.



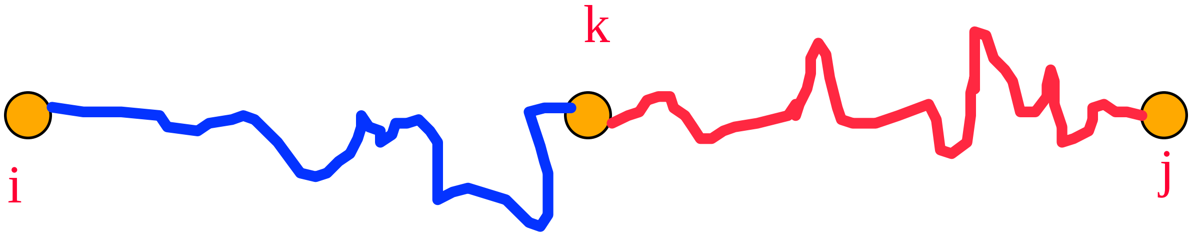
Recurrence For c(i,j,k), k > 0

- The shortest path from vertex i to vertex j that has no intermediate vertex larger than k may or may not go through vertex k.
- If this shortest path does not go through vertex k, the largest permissible intermediate vertex is k-1. So the path length is c(i,j,k-1).



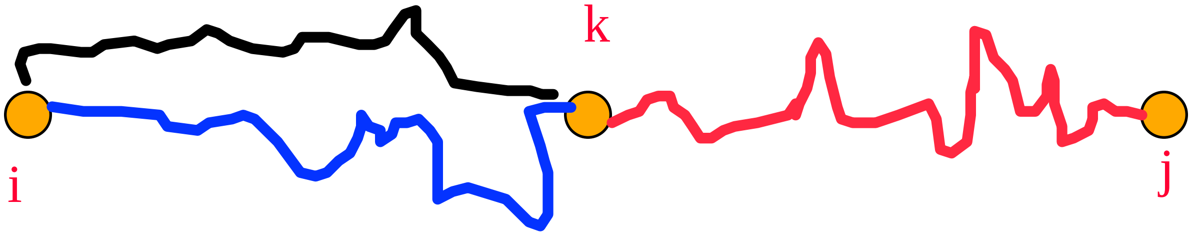
Recurrence For c(i,j,k)), k > 0

- Shortest path goes through vertex k.



- We may assume that vertex k is not repeated because no cycle has negative length.
- Largest permissible intermediate vertex on i to k and k to j paths is k-1.

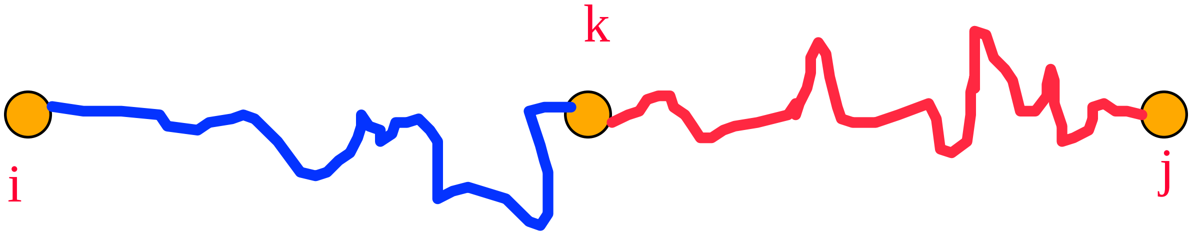
Recurrence For c(i,j,k)), k > 0



- i to k path must be a shortest i to k path that goes through no vertex larger than k-1.

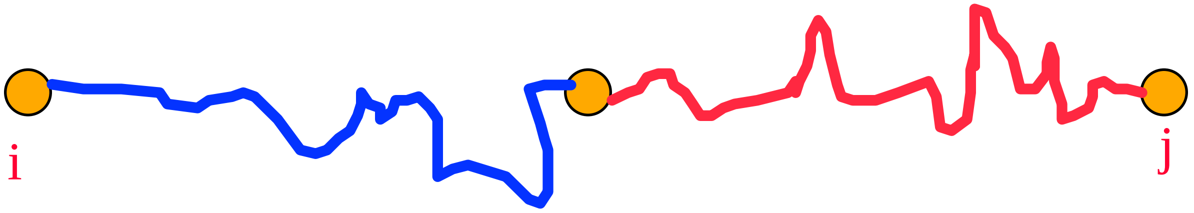
- If not, replace current i to k path with a shorter i to k path to get an even shorter i to j path.

Recurrence For c(i,j,k)), k > 0



- Similarly, k to j path must be a shortest k to j path that goes through no vertex larger than k-1.
- Therefore, length of i to k path is c(i,k,k-1), and length of k to j path is c(k,j,k-1).
- So, c(i,j,k) = c(i,k,k-1) + c(k,j,k-1).

Recurrence For c(i,j,k)), k > 0



- Combining the two equations for c(i,j,k), we get c(i,j,k) = min{c(i,j,k-1), c(i,k,k-1) + c(k,j,k-1)}.
- We may compute the c(i,j,k)s in the order k = 1, 2, 3, …, n.

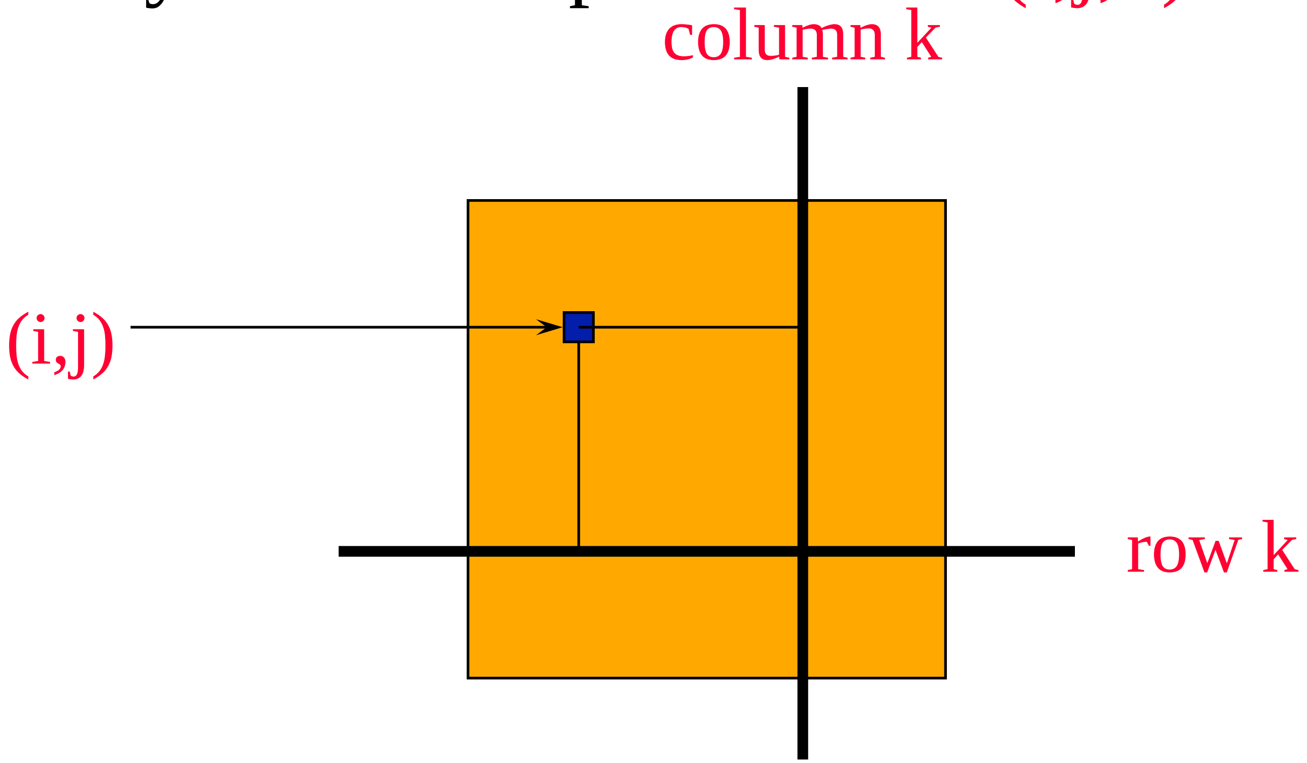
Floyd’s Shortest Paths Algorithm

```
for (int k = 1; k <= n; k++)
   for (int i = 1; i <= n; i++)
      for (int j = 1; j <= n; j++)
          c(i,j,k) = min{c(i,j,k-1),
                                      c(i,k,k-1) + c(k,j,k-1)};
```

- Time complexity is O(n^3).
- More precisely Theta(n^3).
- ^{3}) .

Space Reduction

- $c(i,j,k) = \min\{c(i,j,k-1), c(i,k,k-1) + c(k,j,k-1)\}$
- When neither i nor j equals k, $c(i,j,k-1)$ is used only in the computation of $c(i,j,k)$.



- So $c(i,j,k)$ can overwrite $c(i,j,k-1)$.

Space Reduction

- $c(i,j,k) = \min\{c(i,j,k-1), c(i,k,k-1) + c(k,j,k-1)\}$
- When i equals k, $c(i,j,k-1)$ equals $c(i,j,k)$.
  - $c(k,j,k) = \min\{c(k,j,k-1), c(k,k,k-1) + c(k,j,k-1)\}$

    $= \min\{c(k,j,k-1), 0 + c(k,j,k-1)\}$

    $= c(k,j,k-1)$
- So, when i equals k, $c(i,j,k)$ can overwrite $c(i,j,k-1)$.
- Similarly when j equals k, $c(i,j,k)$ can overwrite $c(i,j,k-1)$.
- So, in all cases $c(i,j,k)$ can overwrite $c(i,j,k-1)$.

Floyd’s Shortest Paths Algorithm

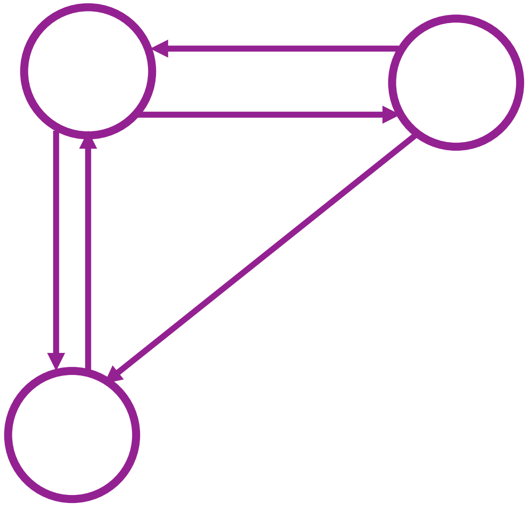
```
for (int k = 1; k <= n; k++)
  for (int i = 1; i <= n; i++)
    for (int j = 1; j <= n; j++)
      c(i,j) = min{c(i,j), c(i,k) + c(k,j)};
```

- Initially, c(i,j) = c(i,j,0).
- Upon termination, c(i,j) = c(i,j,n).
- Time complexity is Theta(n^3).
- Theta(n^2) space is needed for c(*,*).

Building The Shortest Paths

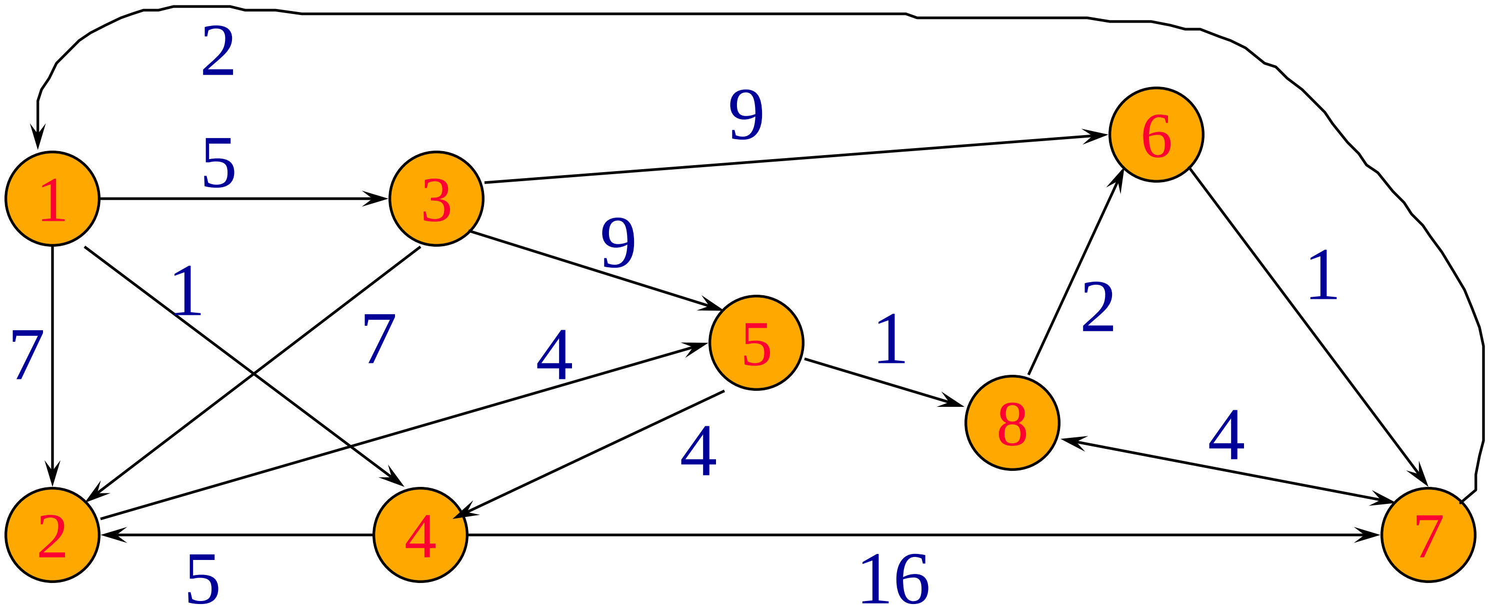
- Let kay(i,j) be the largest vertex on the shortest path from i to j.
- Initially, kay(i,j) = 0 (shortest path has no intermediate vertex).

```
for (int k = 1; k <= n; k++)
  for (int i = 1; i <= n; i++)
    for (int j = 1; j <= n; j++)
      if (c(i,j) > c(i,k) + c(k,j))
      {kay(i,j) = k; c(i,j) = c(i,k) + c(k,j);}
```



∞

Example



$$\begin{matrix} - & 7 & 5 & 1 & - & - & - & - \\ - & - & - & - & 4 & - & - & - \\ - & 7 & - & - & 9 & 9 & - & - \\ - & 5 & - & - & - & - & 16 & - \\ - & - & - & 4 & - & - & - & 1 \\ - & - & - & - & - & - & 1 & - \\ 2 & - & - & - & - & - & - & 4 \\ - & - & - & - & - & 2 & 4 & - \end{matrix}$$

Initial Cost Matrix

$c(*,*) = c(*,*,0)$

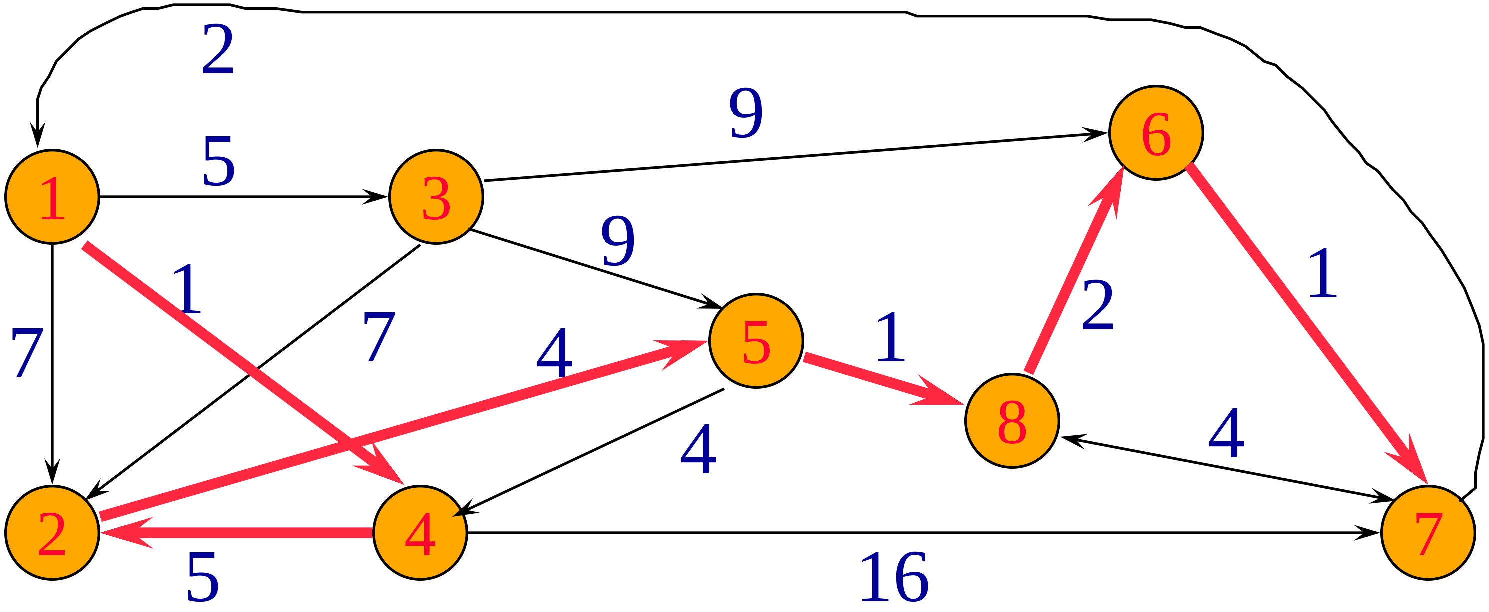
Final Cost Matrix $c(*,*) = c(*,*,n)$

$$\begin{array}{cccccccc}
0 & 6 & 5 & 1 & 10 & 13 & 14 & 11 \\
10 & 0 & 15 & 8 & 4 & 7 & 8 & 5 \\
12 & 7 & 0 & 13 & 9 & 9 & 10 & 10 \\
15 & 5 & 20 & 0 & 9 & 12 & 13 & 10 \\
6 & 9 & 11 & 4 & 0 & 3 & 4 & 1 \\
3 & 9 & 8 & 4 & 13 & 0 & 1 & 5 \\
2 & 8 & 7 & 3 & 12 & 6 & 0 & 4 \\
5 & 11 & 10 & 6 & 15 & 2 & 3 & 0
\end{array}$$

kay Matrix

0	4	0	0	4	8	8	5
8	0	8	5	0	8	8	5
7	0	0	5	0	0	6	5
8	0	8	0	2	8	8	5
8	4	8	0	0	8	8	0
7	7	7	7	7	0	0	7
0	4	1	1	4	8	0	0
7	7	7	7	7	0	6	0

Shortest Path



Shortest path from 1 to 7.

Path length is 14.

Build A Shortest Path

0	4	0	0	4	8	8	5
8	0	8	5	0	8	8	5
7	0	0	5	0	0	6	5
8	0	8	0	2	8	8	5
8	4	8	0	0	8	8	0
7	7	7	7	7	0	0	7
0	4	1	1	4	8	0	0
7	7	7	7	7	0	6	0

- The path is 1 4 2 5 8 6 7.
- kay(1,7) = 8

1 → 8 → 7

- kay(1,8) = 5

1 → 5 → 8 → 7

- kay(1,5) = 4

1 → 4 → 5 → 8 → 7

Build A Shortest Path

0	4	0	0	4	8	8	5
8	0	8	5	0	8	8	5
7	0	0	5	0	0	6	5
8	0	8	0	2	8	8	5
8	4	8	0	0	8	8	0
7	7	7	7	7	0	0	7
0	4	1	1	4	8	0	0
7	7	7	7	7	0	6	0

- The path is 1 4 2 5 8 6 7.

1 → 4 → 5 → 8 → 7

- kay(1,4) = 0

1 4 → 5 → 8 → 7

- kay(4,5) = 2

1 4 → 2 → 5 → 8 → 7

- kay(4,2) = 0

1 4 2 → 5 → 8 → 7

Build A Shortest Path

0	4	0	0	4	8	8	5
8	0	8	5	0	8	8	5
7	0	0	5	0	0	6	5
8	0	8	0	2	8	8	5
8	4	8	0	0	8	8	0
7	7	7	7	7	0	0	7
0	4	1	1	4	8	0	0
7	7	7	7	7	0	6	0

- The path is 1 4 2 5 8 6 7.

1 4 2 → 5 → 8 → 7

- kay(2,5) = 0

1 4 2 5 → 8 → 7

- kay(5,8) = 0

1 4 2 5 8 → 7

- kay(8,7) = 6

1 4 2 5 8 → 6 → 7

Build A Shortest Path

0	4	0	0	4	8	8	5
8	0	8	5	0	8	8	5
7	0	0	5	0	0	6	5
8	0	8	0	2	8	8	5
8	4	8	0	0	8	8	0
7	7	7	7	7	0	0	7
0	4	1	1	4	8	0	0
7	7	7	7	7	0	6	0

- The path is 1 4 2 5 8 6 7.

  1 4 2 5 8 → 6 → 7

- kay(8,6) = 0

  1 4 2 5 8 6 → 7

- kay(6,7) = 0

  1 4 2 5 8 6 7

Output A Shortest Path

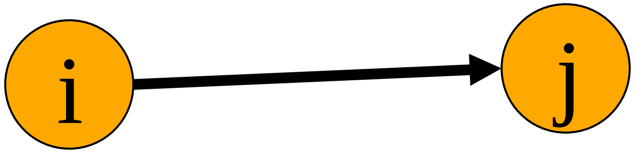
```
void outputPath(int i, int j)
{// does not output first vertex (i) on path
   if (i == j) return;
   if (kay[i][j] == 0)  // no intermediate vertices on path
            print(j + " ");
   else {// kay[i][j] is an intermediate vertex on the path
             outputPath(i, kay[i][j]);
             outputPath(kay[i][j], j);
         }
}
```

Time Complexity Of outputPath

O(number of vertices on shortest path)

Directed Graphs Usage

- Directed graphs are often used to represent order -dependent tasks
- Cannot start a task before another task finishes
- Model this task dependent constraint using *arcs*
- An *arc (i,j)* means *task j* cannot start until *task i* is finished



Task **j** cannot start

until task **i** is finished

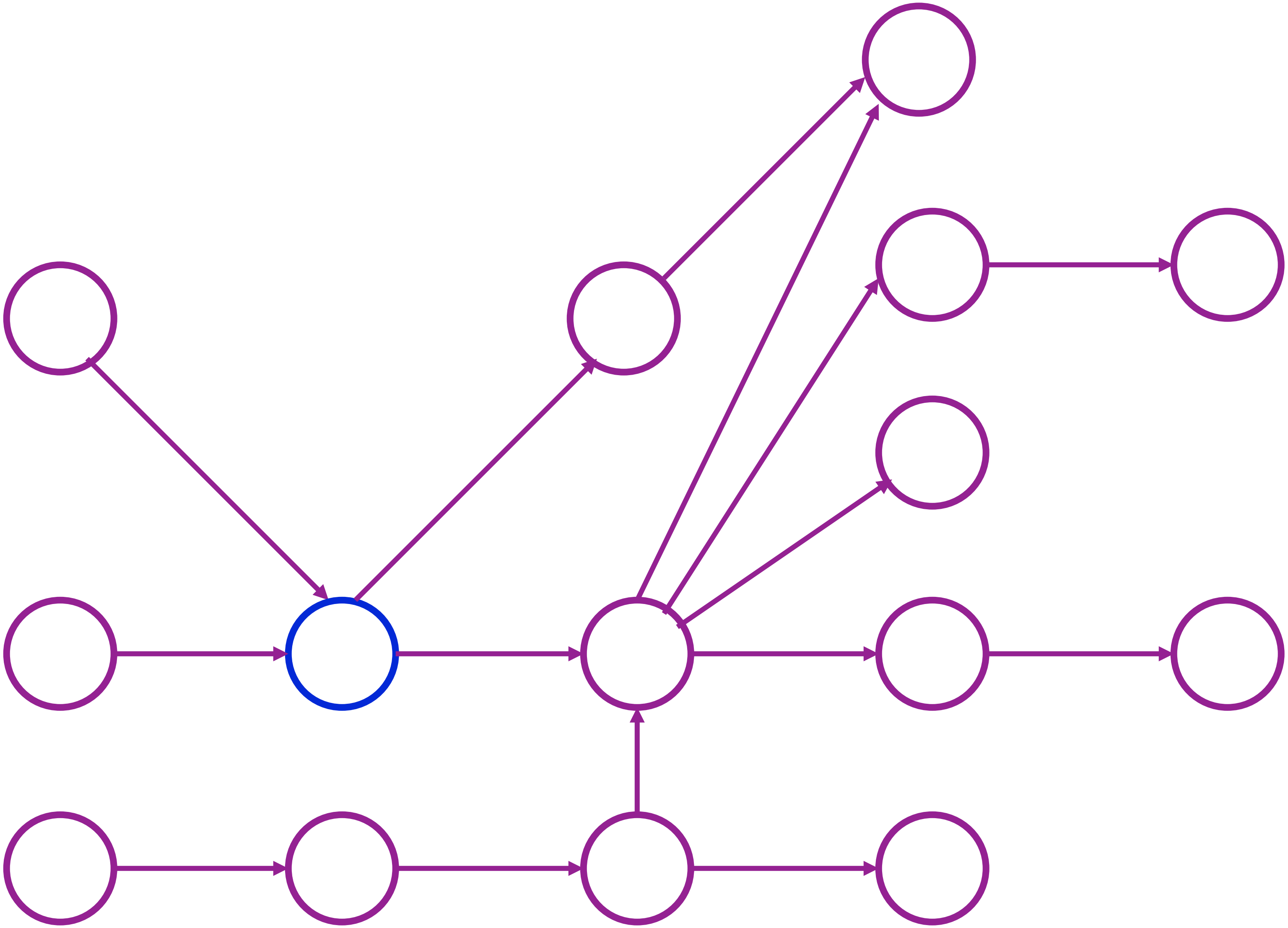
- For the system not to hang, the graph must be acyclic.

Activity Networks

Activity-on-Vertex (AOV) Networks

- A directed graph G
- Vertices
  - Tasks or activities
- Edges
  - Precedence relations between tasks

AOV



Definitions

- Vertex i in an AOV network G is a predecessor of j iff there is a directed path from i to j. If <i, j> is an edge in G then i is an immediate predecessor of j and j immediate successor of i.

- A precedence relation that is both transitive and irreflexive is a partial order.

- A directed graph with no cycle is an acyclic graph.

Problem

- Given an AOV network G
  - whether or not it is irreflexive, i.e., acyclic.
- Solution
  - Generate the **topological order** of

Topological order

- A topological order is a linear ordering of vertices of a graph
  - For any two vertices i and j, if i is a predecessor of j in the network, then i precedes j in the linear ordering
- It can be thought of as a way to linearly order the vertices so that the linear order respects the ordering relations implied by the arcs(edges)

Topological order

Whether a Digraph is acyclic?

- Same to:
  - Does every task can be executed?
- Idea:
  - Tasks have no predecessor can be executed
  - Tasks with all predecessors finished can be executed
  - Starting point must have zero indegree!
  - If it doesn’t exist, the graph would not be acyclic

Whether a Digraph is acyclic?

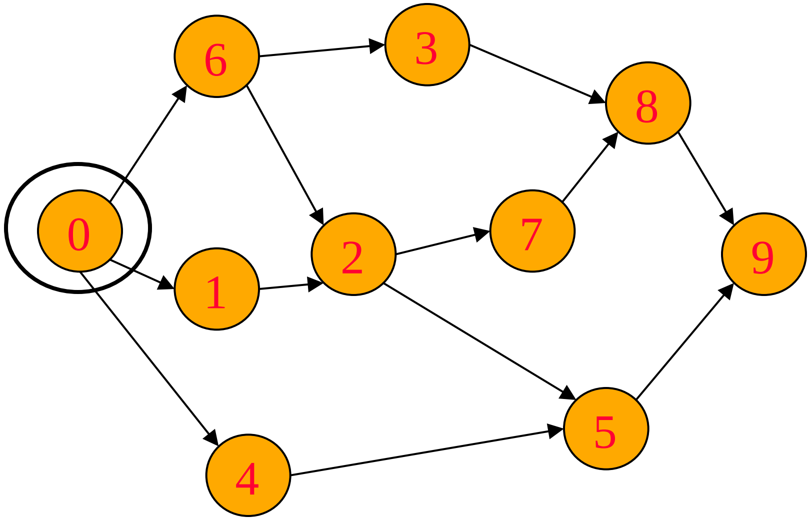
- Vertices with zero *indegree*
  - Can start right away
  - Output it first in the linear order
- A vertex *i* is output
  - Its outgoing arcs *(i, j)* are no longer useful
  - Since tasks *j* does not need to wait for *i* anymore
    - Remove all *i*'s outgoing arcs
- Vertex *i* removed
  - new graph is still a directed acyclic graph
- Repeat step 1-2 until no 0-indegree vertex left

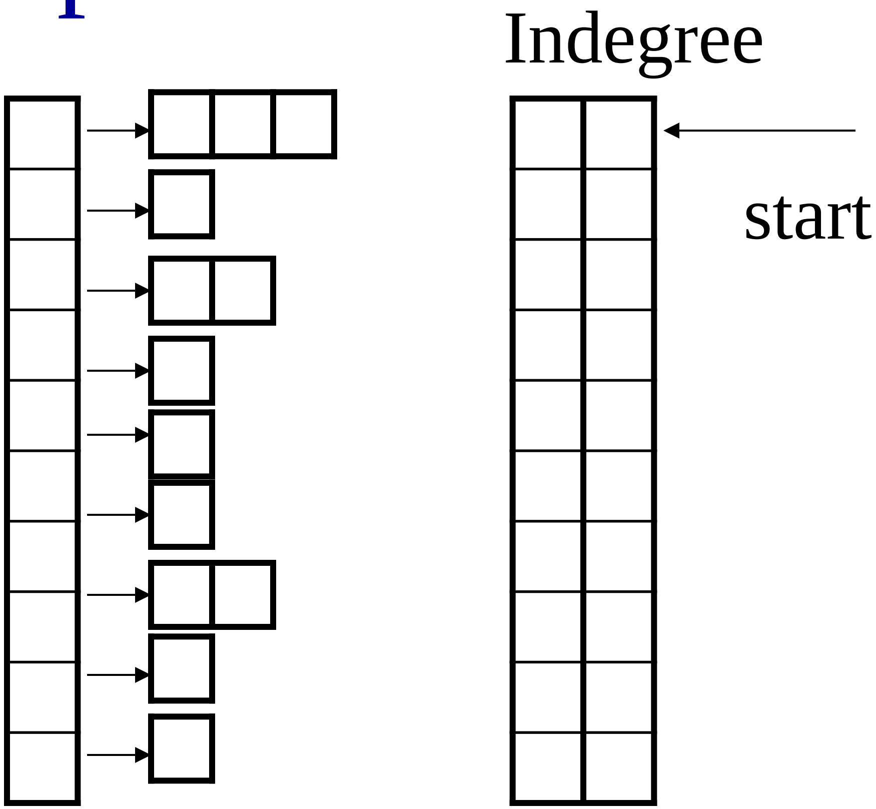
Topological Sort

Algorithm $TSort(G)$

[redacted]

Example

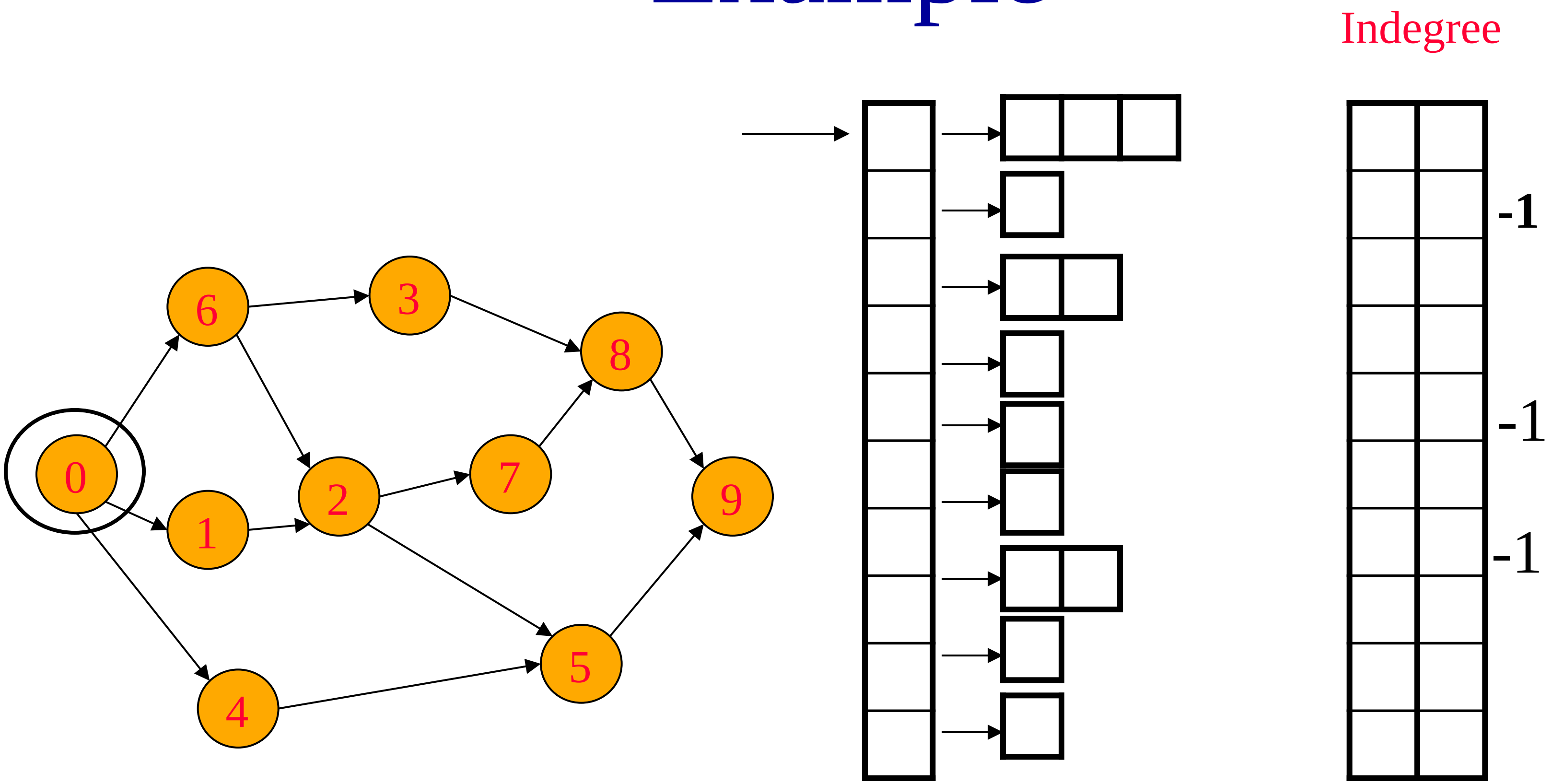




Q = { 0 }

OUTPUT: 0

Example

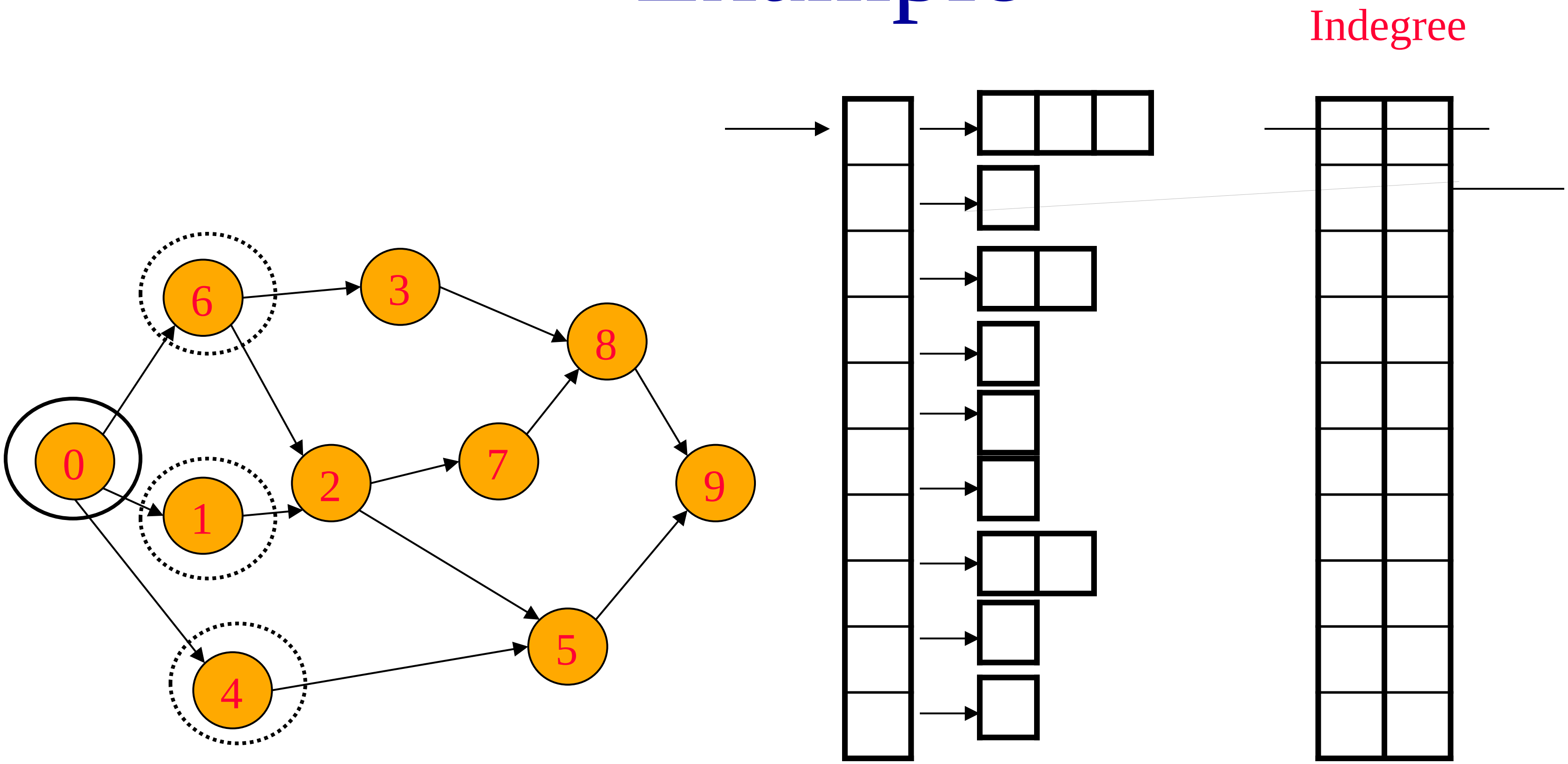


Dequeue 0 Q = { }
-> remove 0’s arcs – adjust indegrees of neighbors

Decrement 0’s neighbors

OUTPUT:

Example



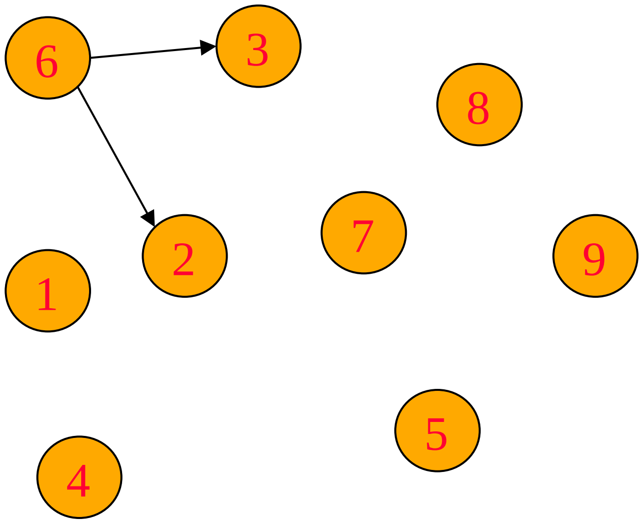
Dequeue 0   Q = { 6, 1, 4 }

Enqueue all starting points

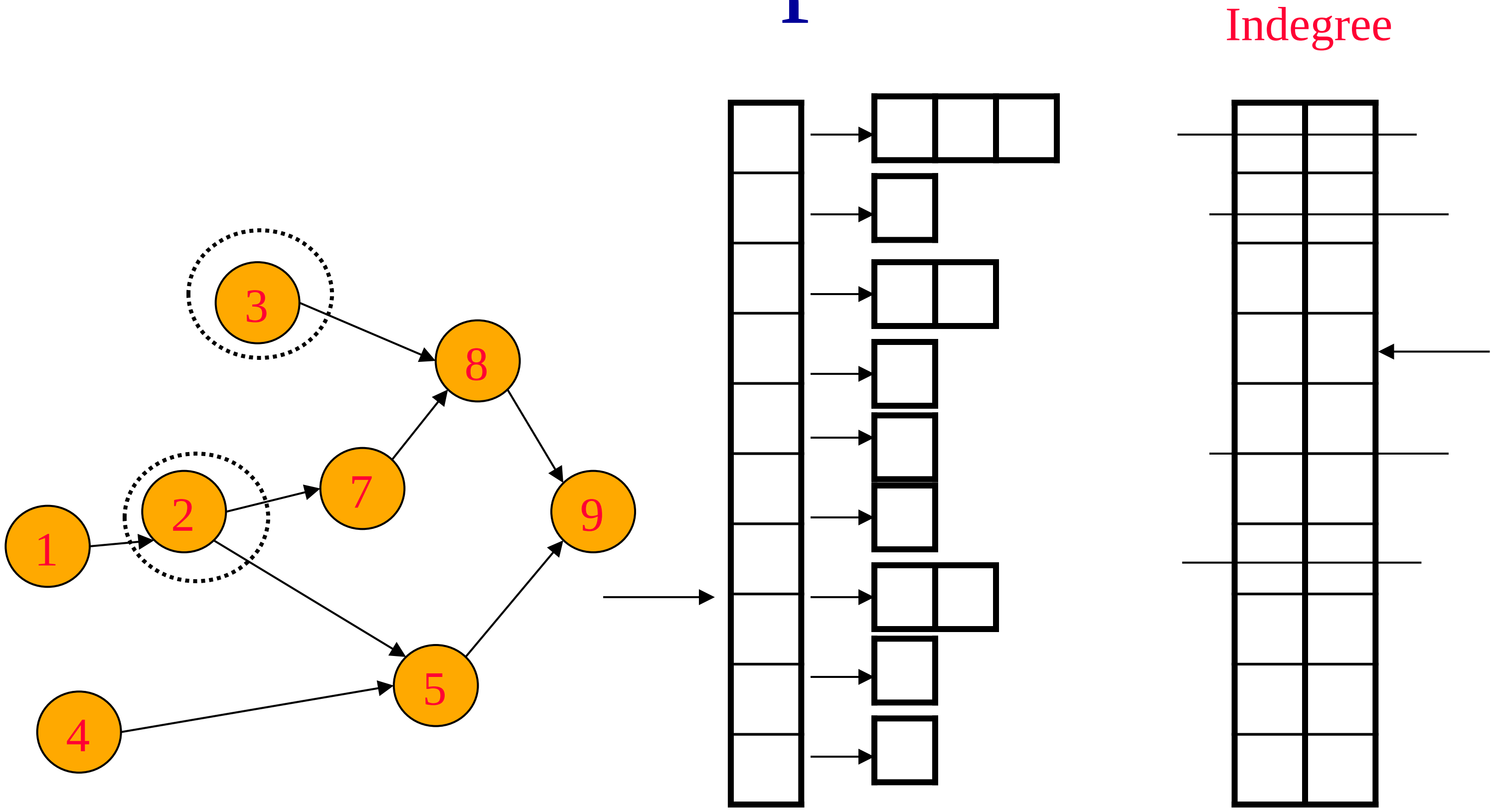
Enqueue all
new start points

OUTPUT: 0

Example



Example

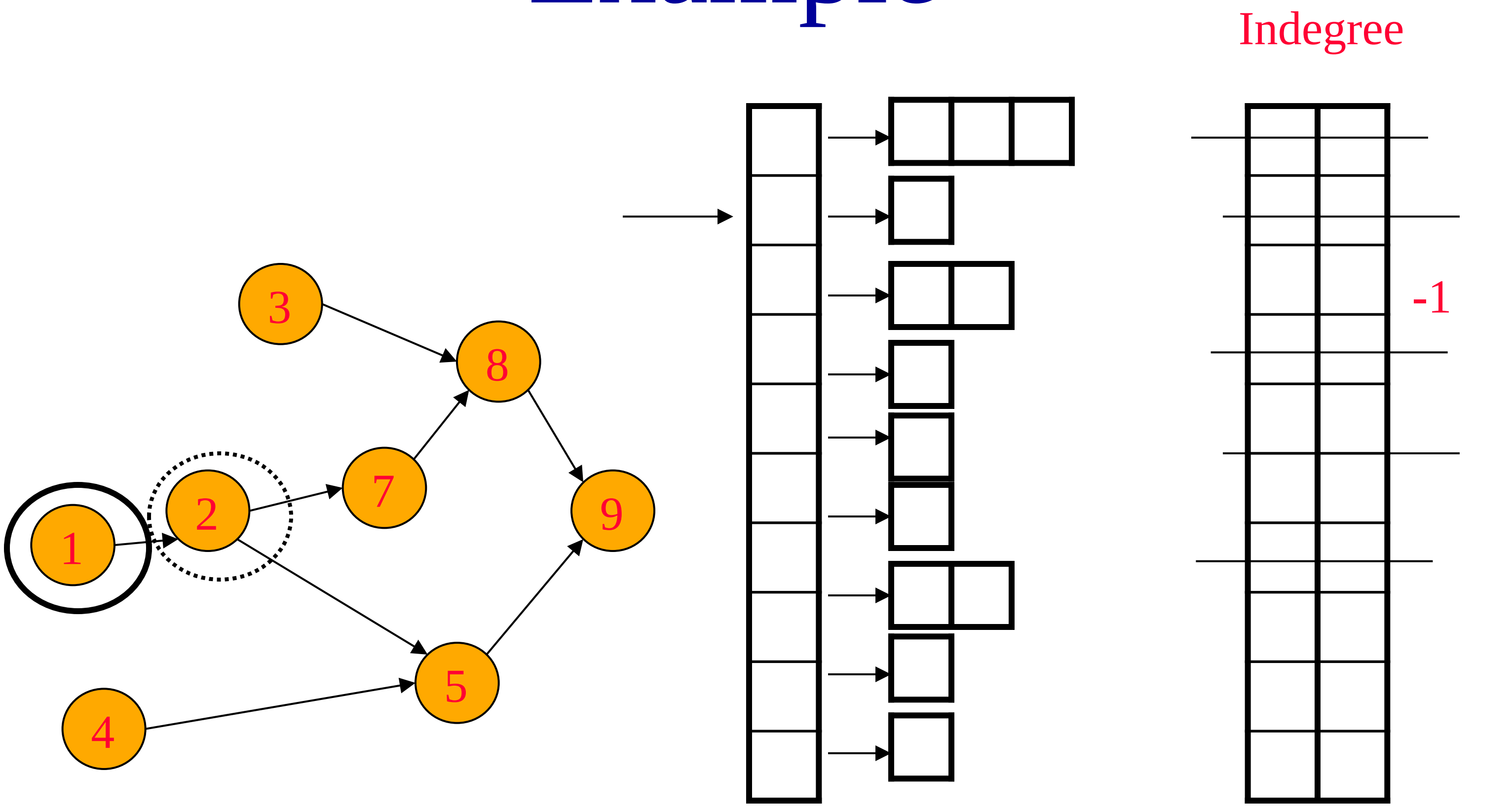


Dequeue 6 Q = { 1, 4, 3 }
Enqueue 3

Enqueue new start

OUTPUT: 0 6

Example

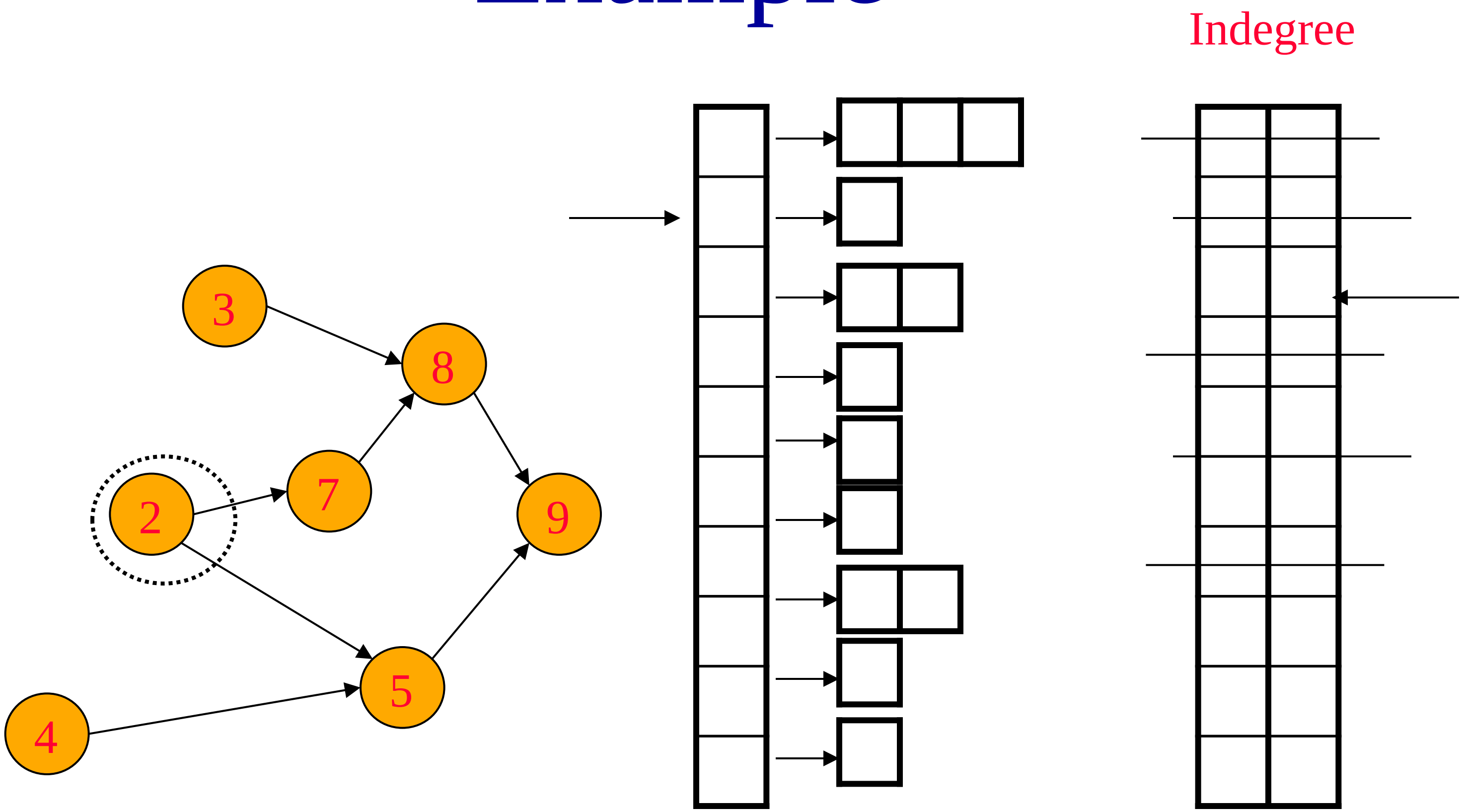


Deque 1 Q = { 4, 3 }
Adjust indegrees of neighbors

Adjust neighbors of 1

OUTPUT: 0 6 1

Example

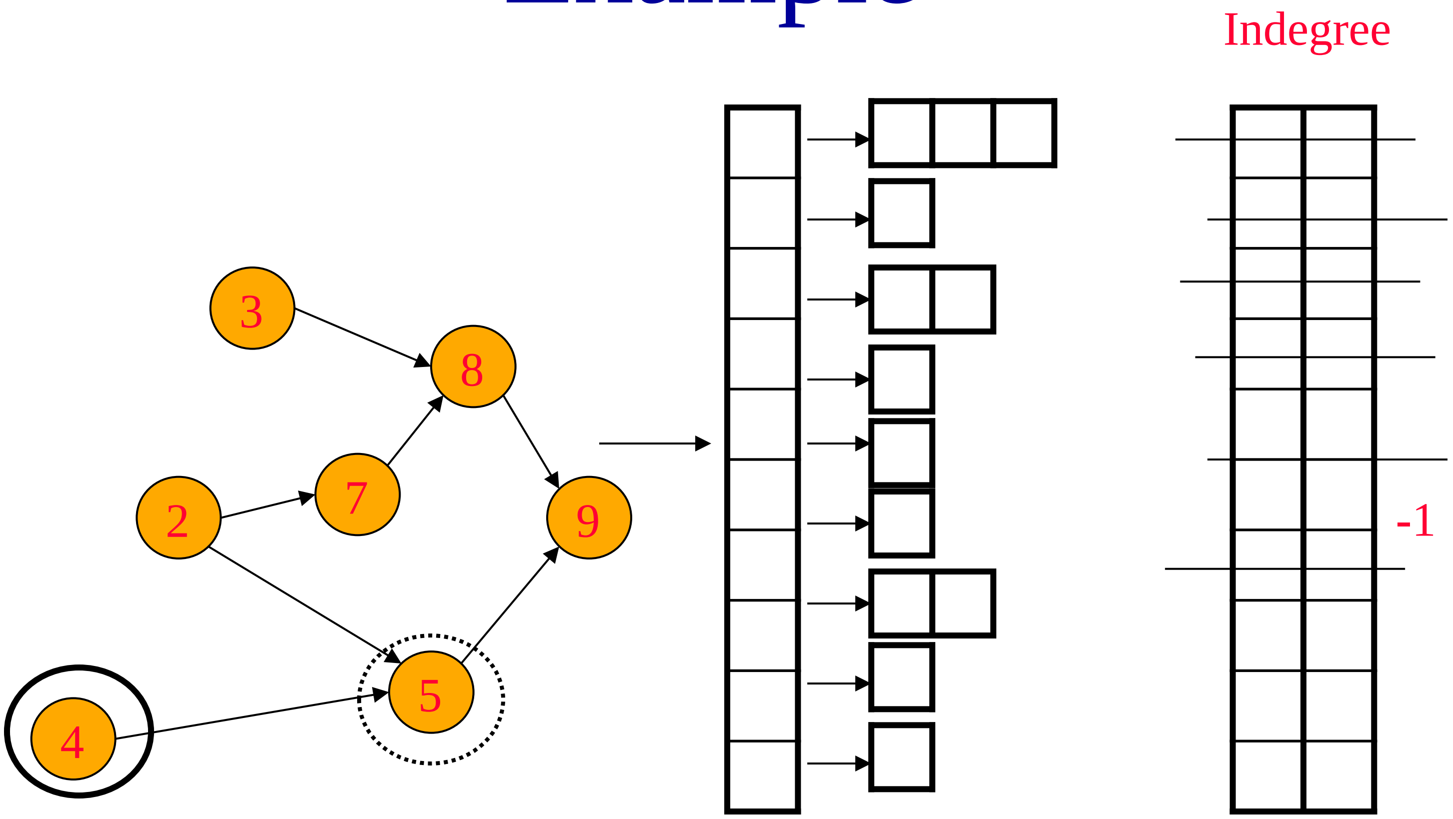


Dequeue 1 Q = { 4, 3, 2 }
Enqueue 2

Enqueue new
starting points

OUTPUT: 0 6 1

Example

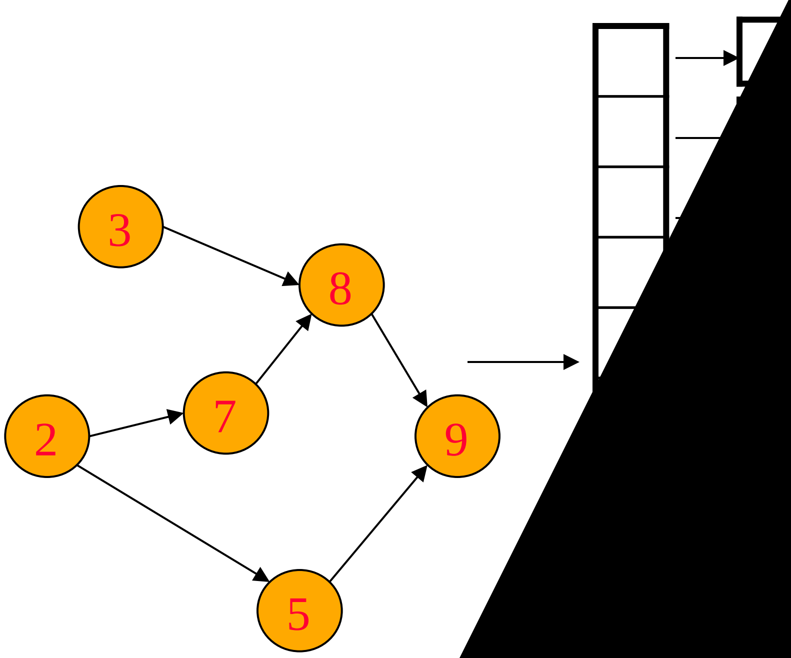


Dequeue 4 Q = { 3, 2 }
Adjust indegrees of neighbors

Adjust 4’s neighbors

OUTPUT: 0 6 1 4

Example

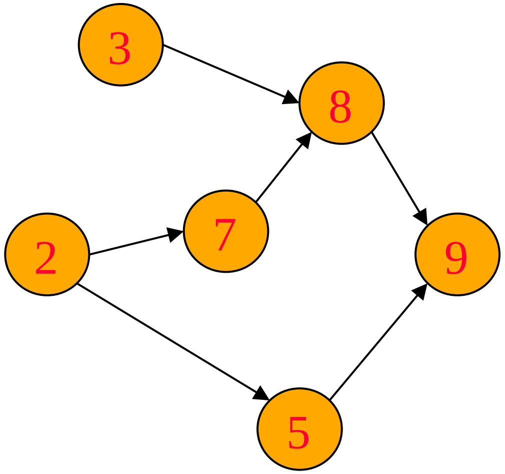


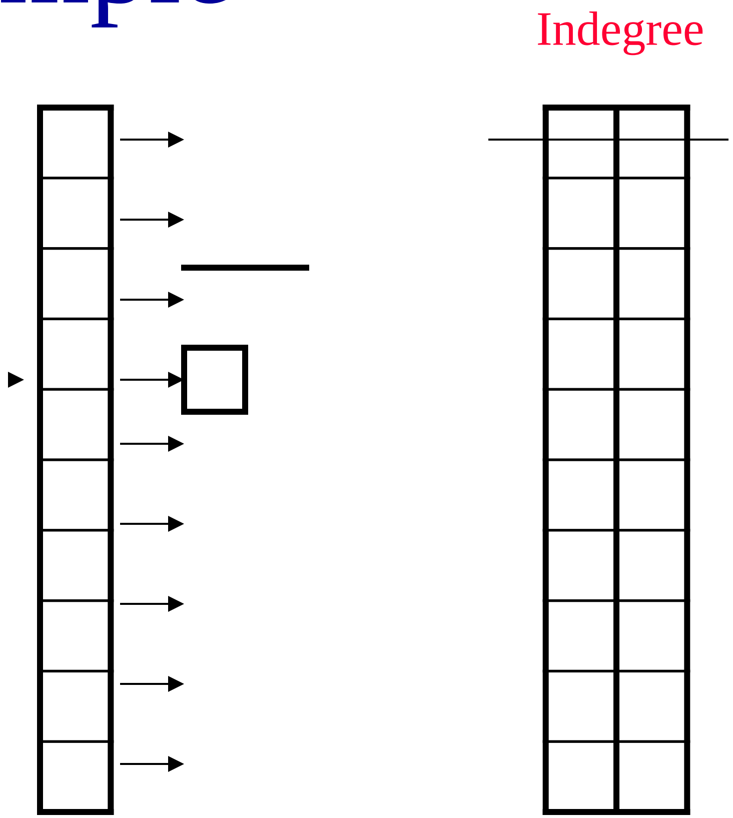
Deque 4 Q = { 3, 2 }
No new start points found

[redacted]

OUTPUT: 0 6 1 4

Example



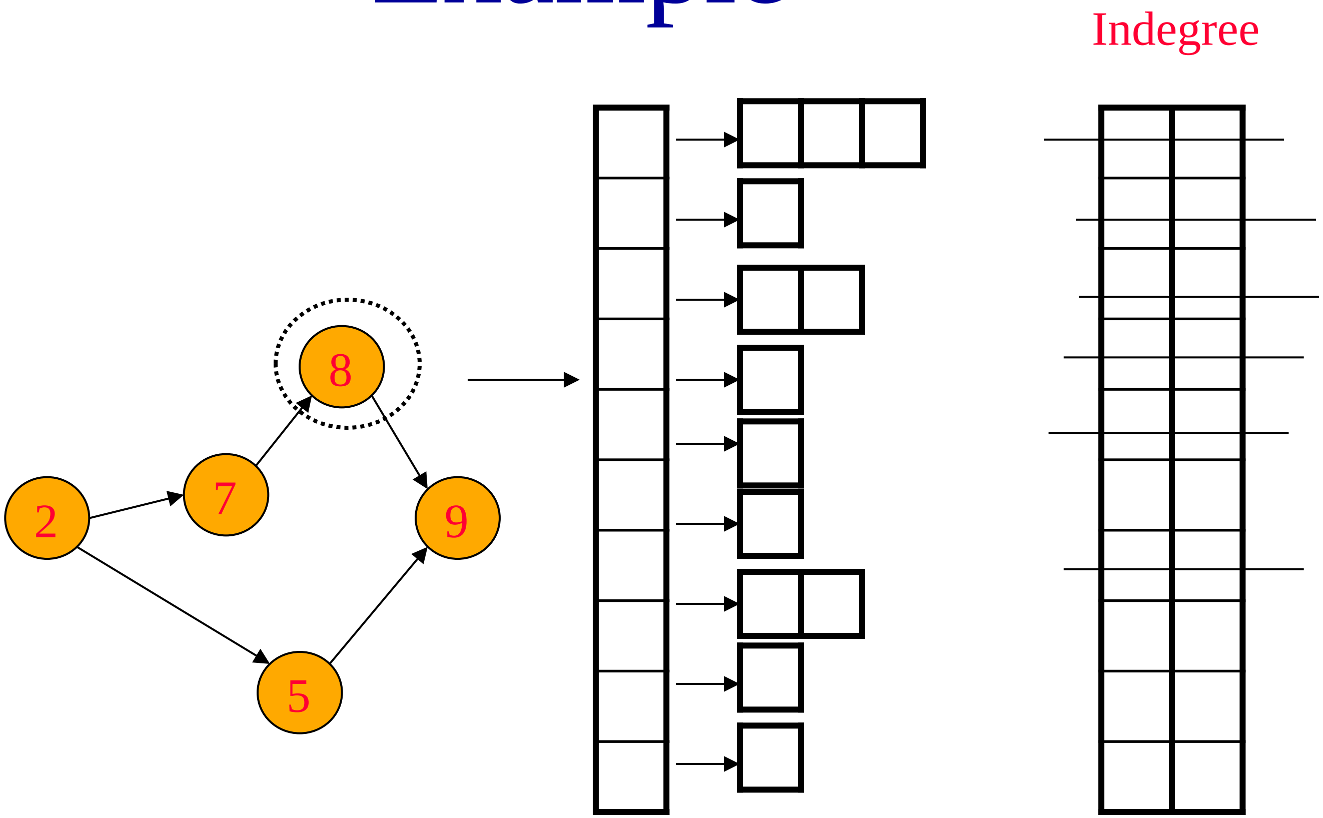


Dequeue 3 Q = { 2 }

Adjust 3's neighbors

OUTPUT: 0 6 1 4 3

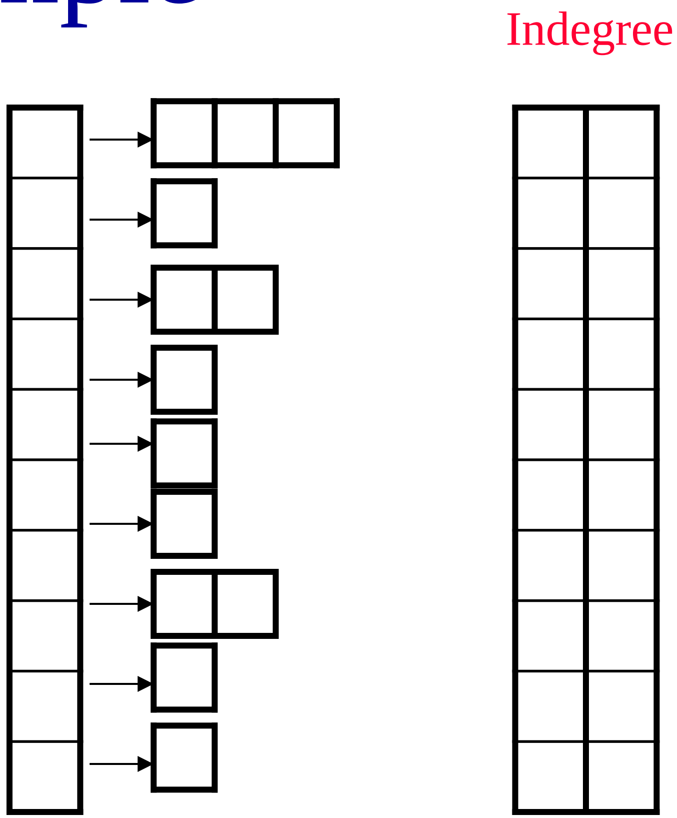
Example

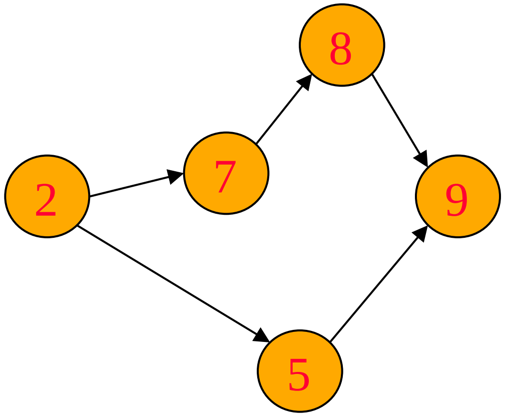


Deque 3 Q = { 2 }
No new start points found

OUTPUT: 0 6 1 4 3

Example

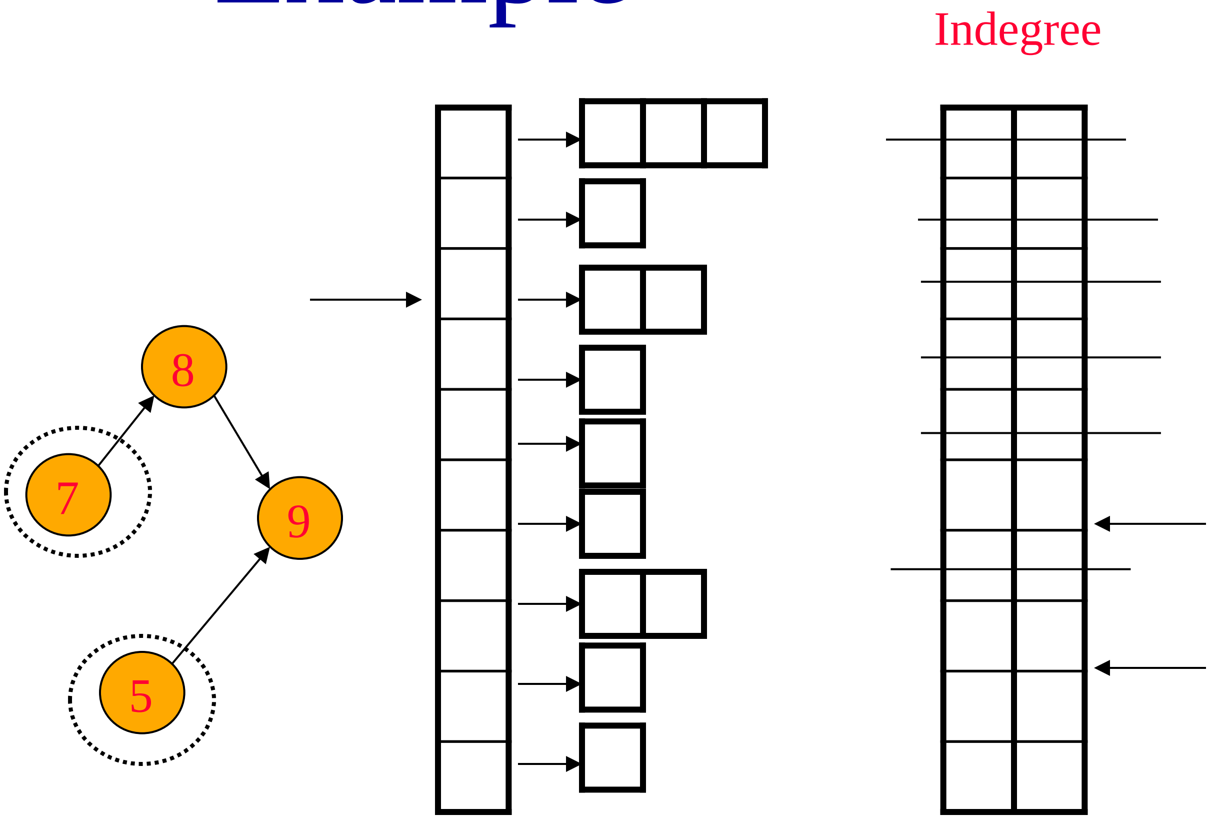




Deque 2 Q = { }

Adjust 2's neighbors

Example

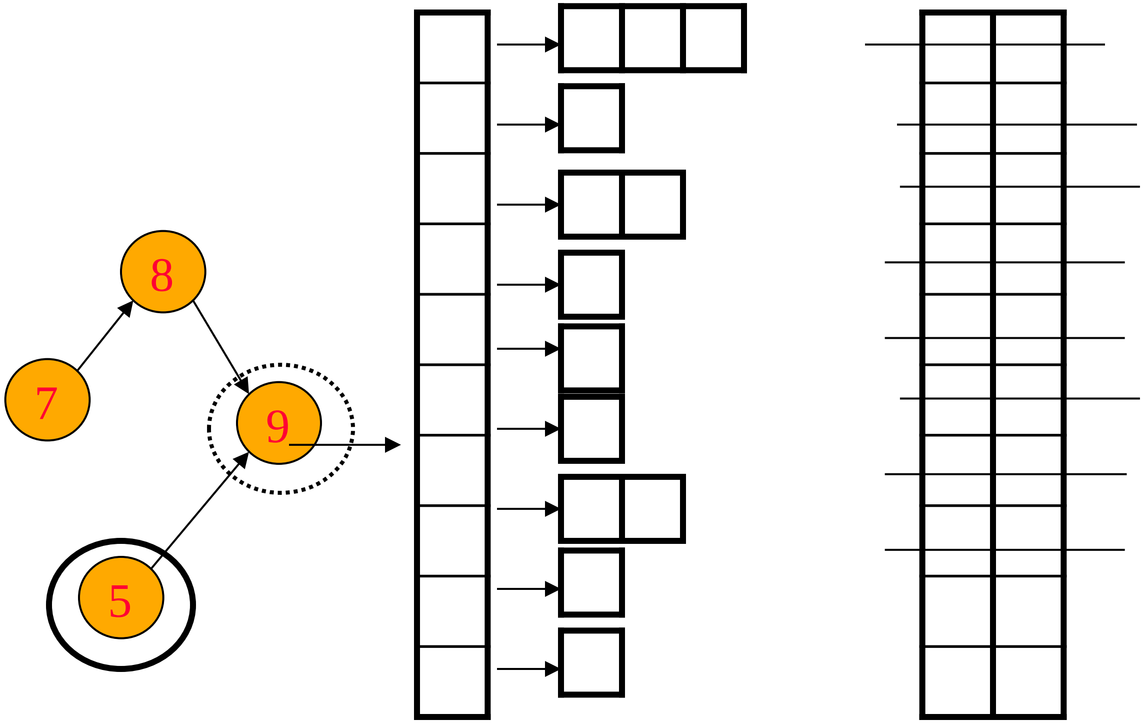


Dequeue 2 Q = { 5, 7 }
Enqueue 5, 7

OUTPUT: 0 6 1 4 3 2

Example

Indegree

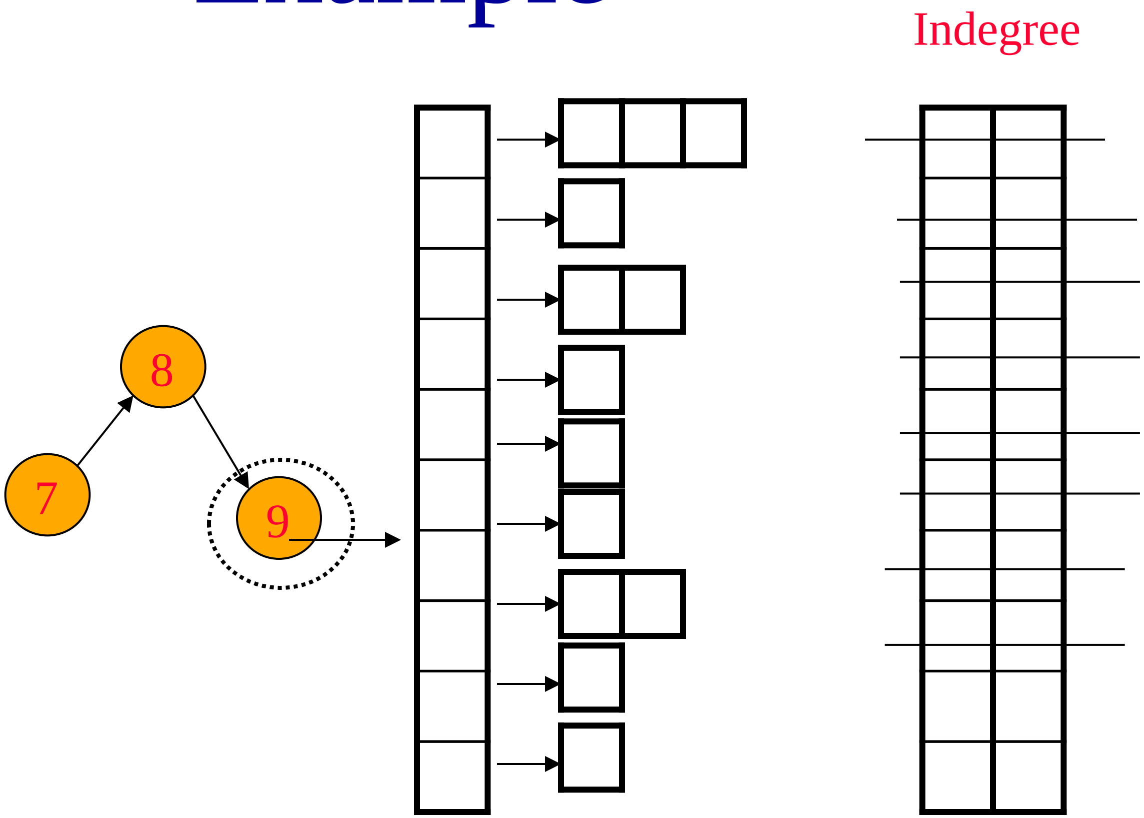


-1

Dequeue 5 Q = { 7 }
Adjust neighbors

OUTPUT: 0 6 1 4 3 2 5

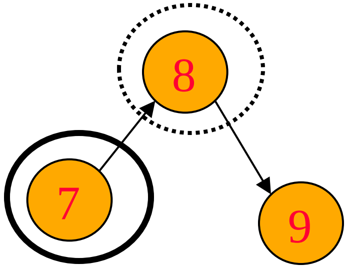
Example

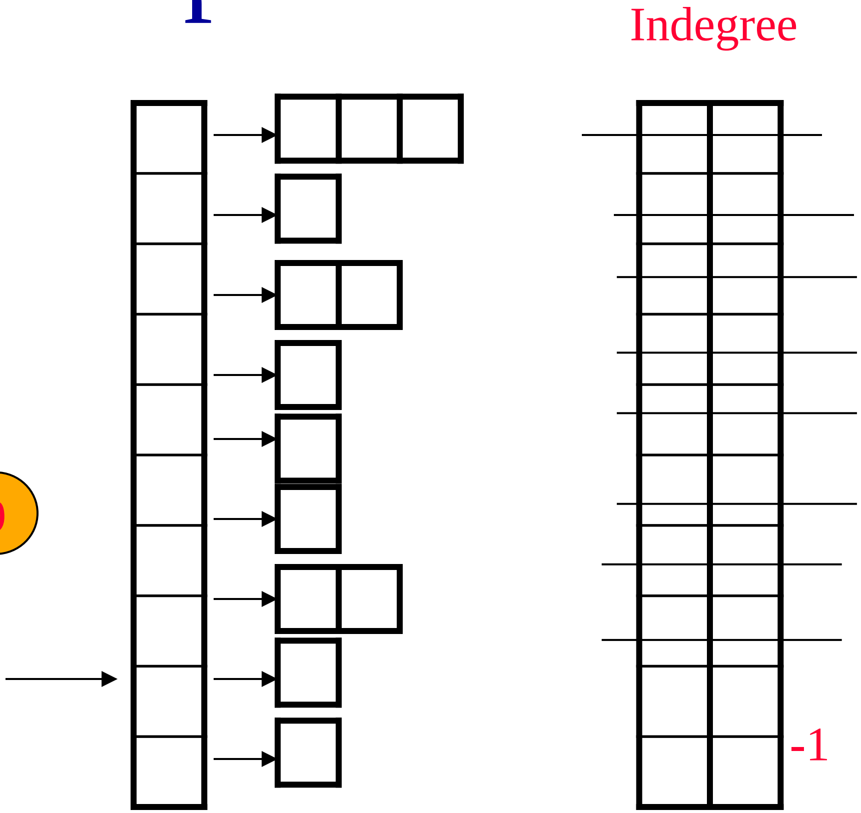


Deque 5 Q = { 7 }
No new starts

OUTPUT: 0 6 1 4 3 2 5

Example



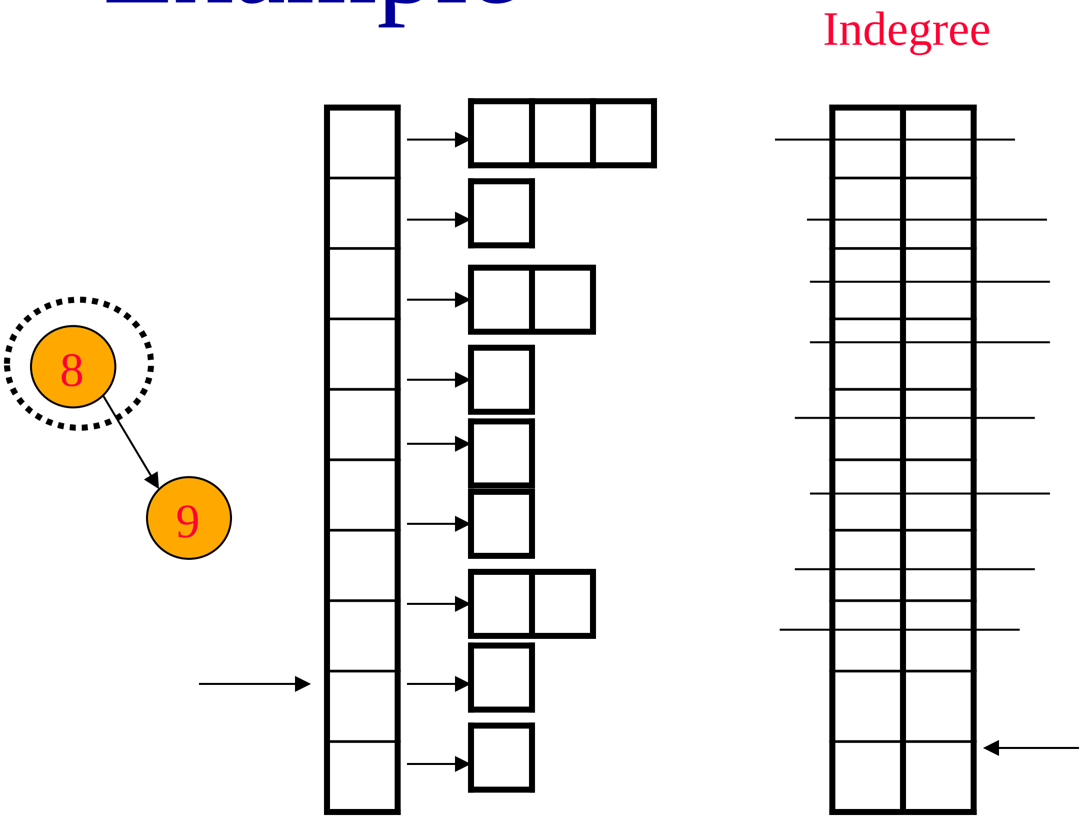


Deqeue 7 Q = { }

Adjust neighbors

OUTPUT: 0 6 1 4 3 2 5 7

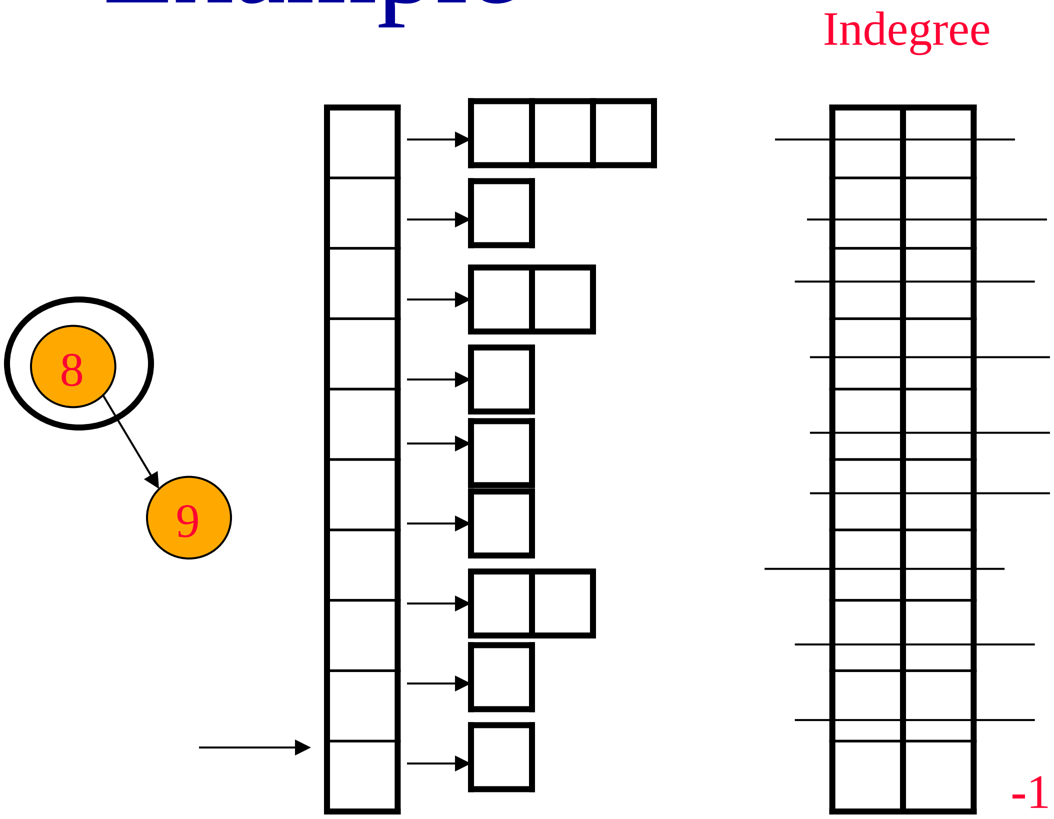
Example



Dequeue 7 Q = { 8 }
Enqueue 8

OUTPUT: 0 6 1 4 3 2 5 7

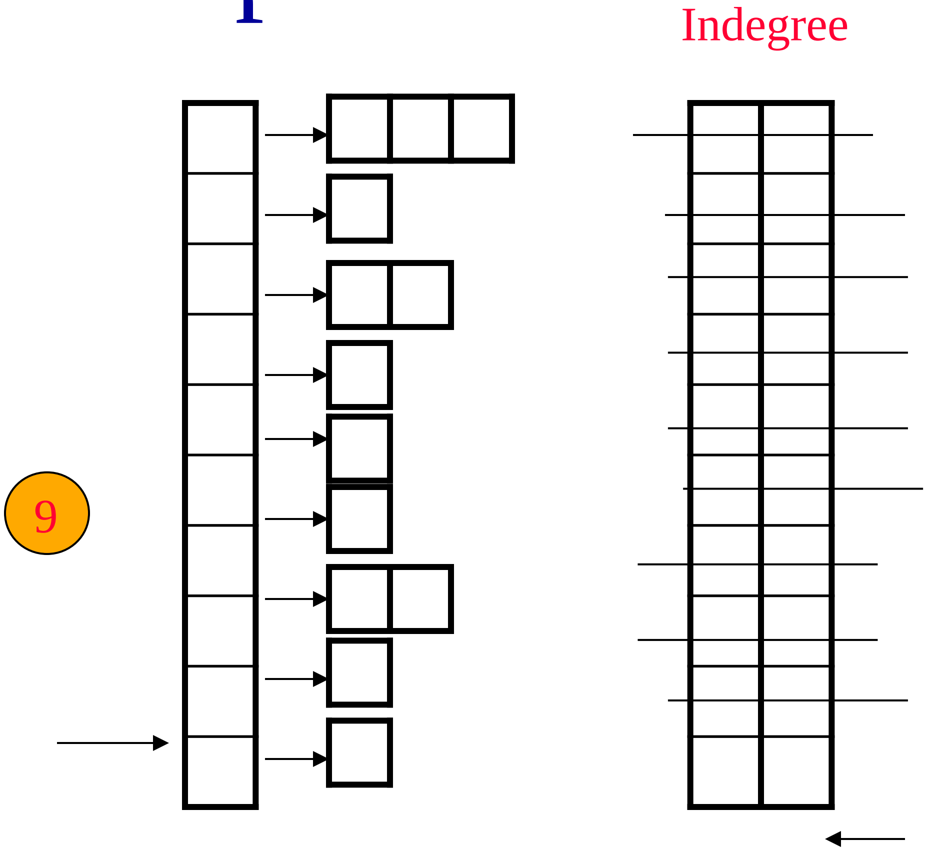
Example



Dequeue 8 Q = { }
Adjust indegrees of neighbors

OUTPUT: 0 6 1 4 3 2 5 7 8

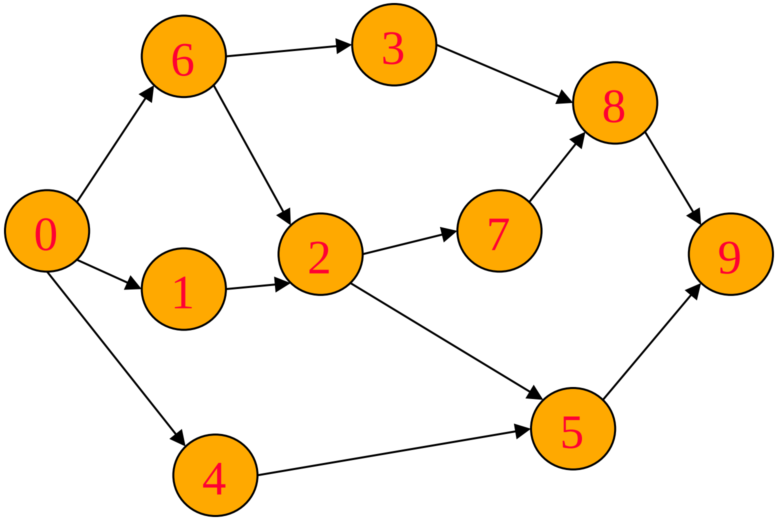
Example



Dequeue 8 Q = { 9 }
Enqueue 9
Dequeue 9 Q = { }
STOP – no neighbors

OUTPUT: 0 6 1 4 3 2 5 7 8 9

Example



OUTPUT: 0 6 1 4 3 2 5 7 8 9

Is output topologically correct?

Topological Sort: Complexity

- We never visited a vertex more than one time

- For each vertex, we had to examine all outgoing edges
  - $\sum outdegree(v) = m$
  - This is summed over all vertices, not per vertex
- So, our running time is exactly
  - O(n + m)
- Can we use a stack instead of a queue?

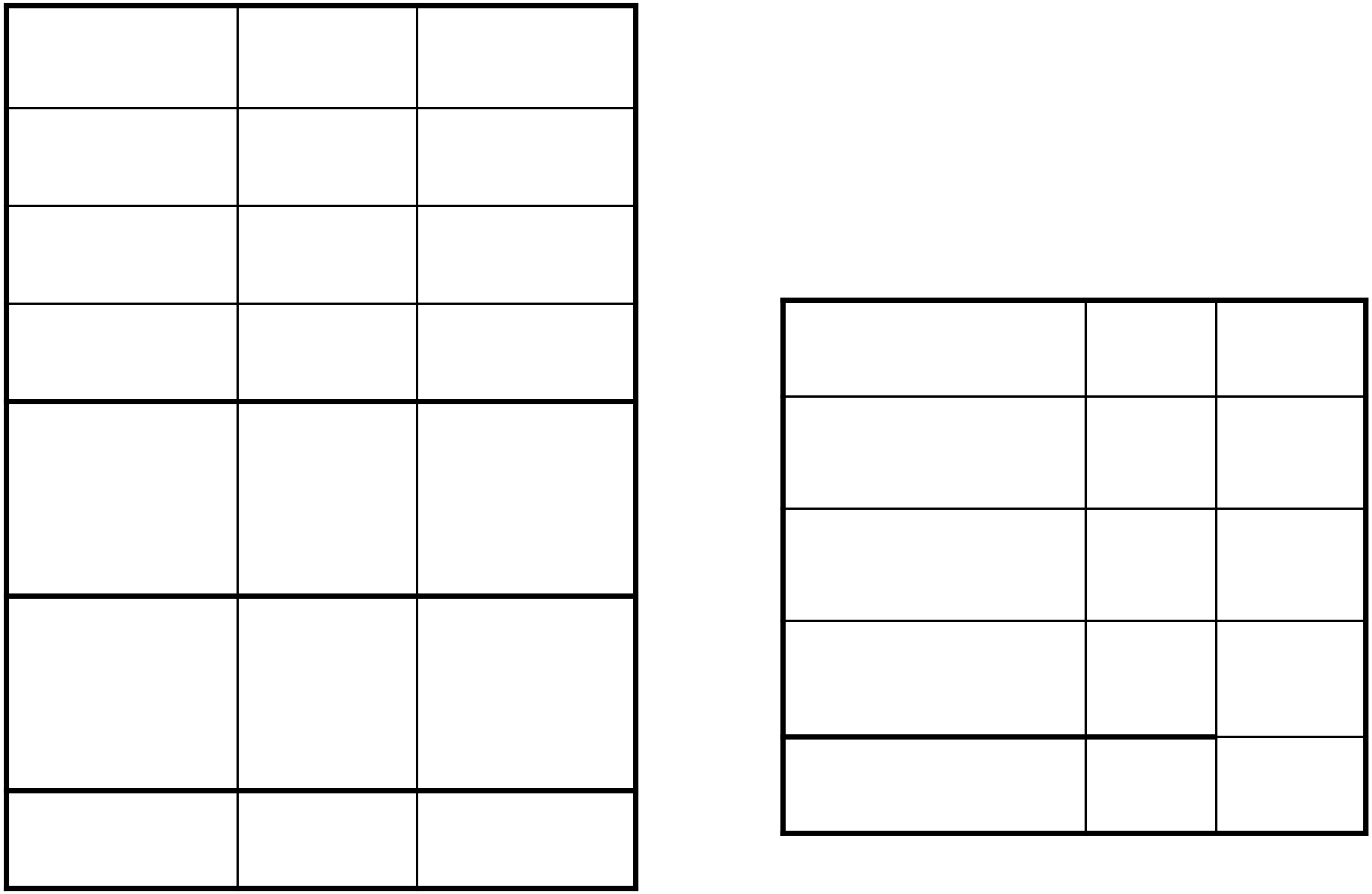
```
Input the AOV network, let n be the number of vertices;
for (int i=0; i<n; i++) // output the vertices
{
   if (every vertex has a predecessor) return;
      // network has a cycle and is infeasible.
   pick a vertex v that has no predecessors;
   cout << v;
   delete v and all edges leading out of v from the network;
}
```

```
• void LinkedGraph::TopologicalOrder() { // count[i] = indegree(i)
•    int top = -1, pos = 0;
•   for (int i=0; i<n; i++) //create a linked stack of vertices with
•      if (count[i]==0) { count[i]=top;  top=i;} //no predecessors
•   for (i=0; i<n; i++)
•      if (top==-1) throw “network has a cycle.  ;
•     int j=top;  top=count[top]; //unstack a vertex
•          t[pos++] = j;  // store vertex j in topological order
•     Chain<int>::ChainIterator ji=adjLists[j].begin();
•    while (ji != adjLists[j].end()) { // decrease the count of
•        count[*ji]--;                    // the successor vertices of j
•         if (count[*ji]==0) {count[*ji]=top; top=*ji;} //add to stack
•        ji++; // next successor
•    }
• }
```

Project Planning Problem

- A project
  - Several tasks
  - Task time
  - Task dependencies
- Problem
  - How long at least to finish the project (all tasks)?
  - What tasks are critical to the finish time?

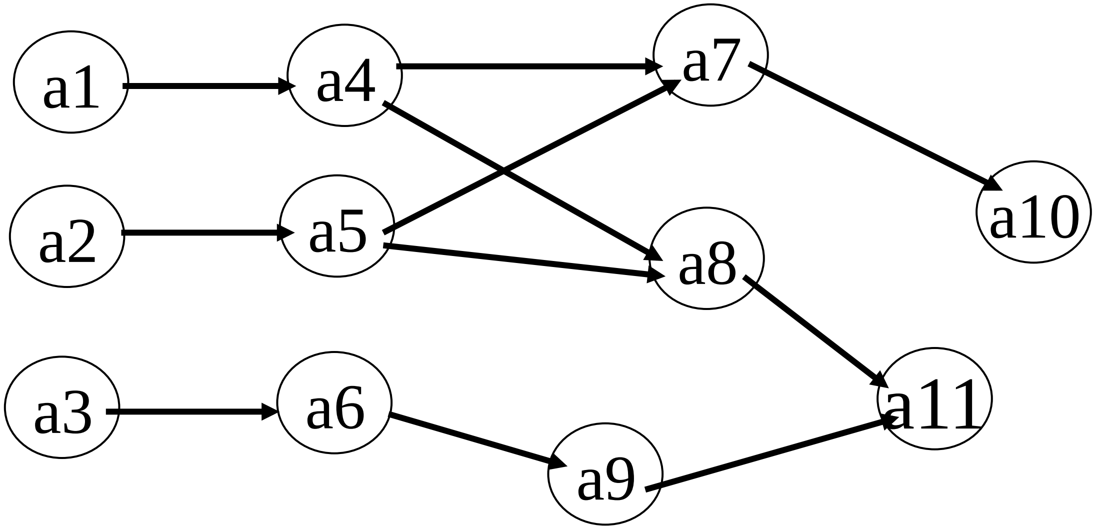
An example



Problem Analysis

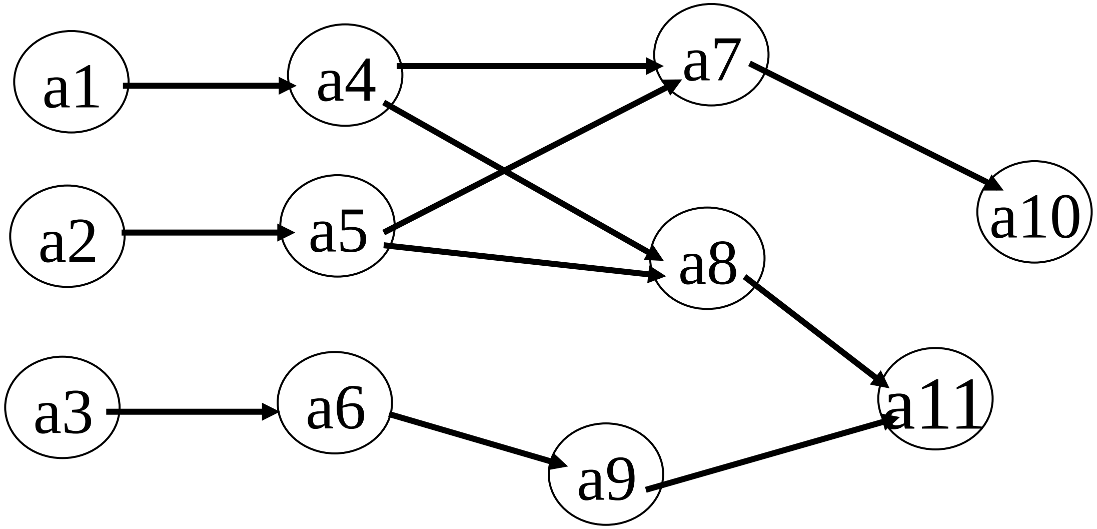
- Problem
  - How long at least to finish the project (all tasks)?
  - What tasks are critical to the finish time?
- Key words
  - At Least
    - No delay
  - Critical
    - Delay is not allowed

AOV



- Problem
  - How long at least to finish the project (all tasks)?
  - What tasks are critical to the finish time?

Possible Solution



- Topological Sort on AOV?
  - Output task
  - Does not know whether the project is finished or not

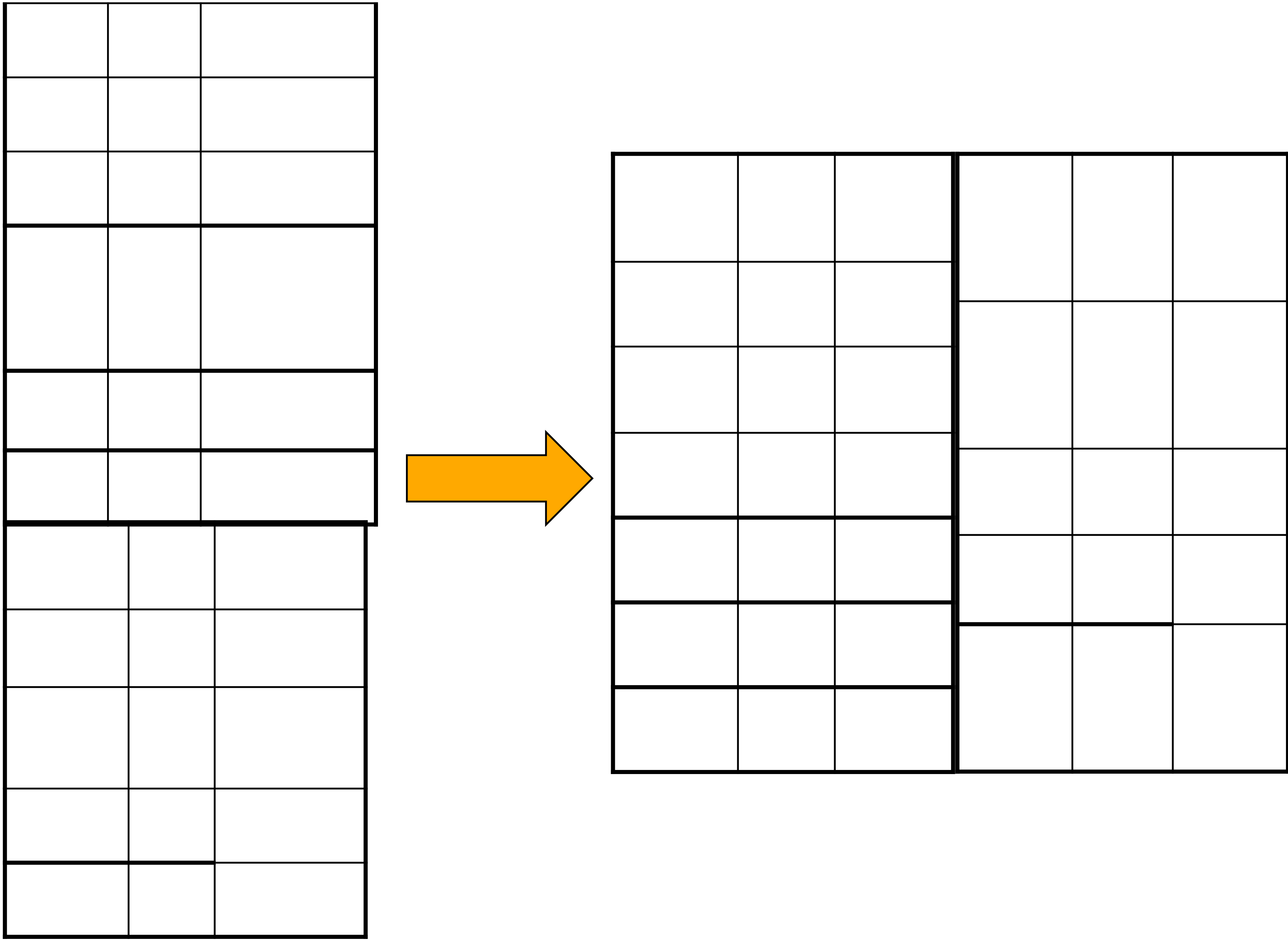
Possible Solution

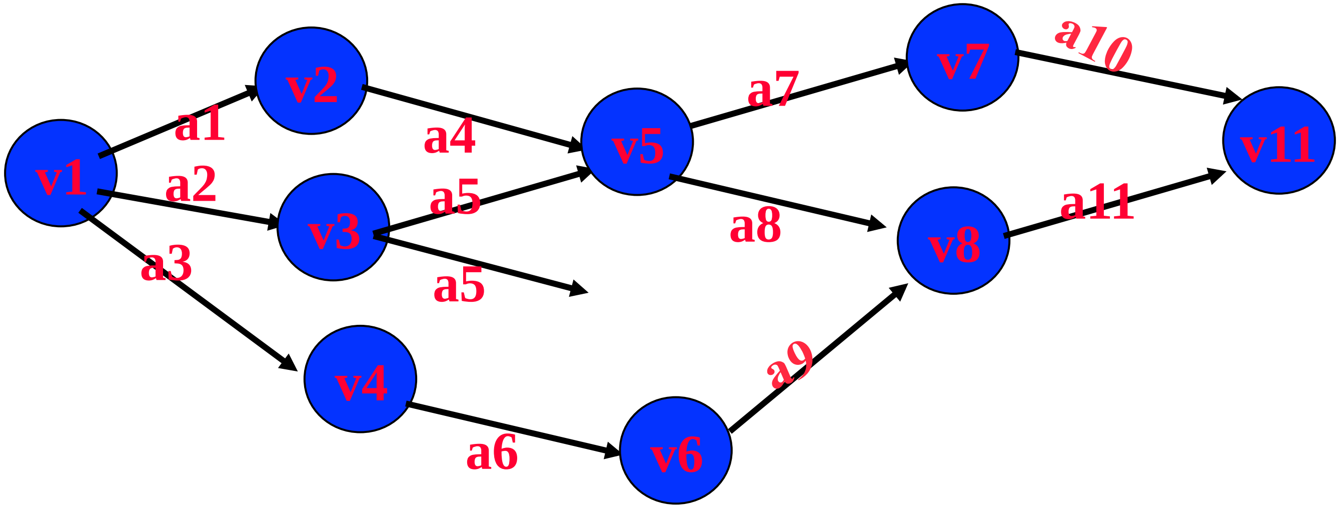
- Analysis
  - We should know what tasks are finished at a given time point
  - Time point
    - Project Phase
    - E.g : after phase 1, task1, 2, 3 are finished

      after phase 2, task1, 2, 3,4,5,6 are finished

Possible Solution

- If the outputs of topological sort are project phases…
  - We did it!
- How to make it happen
  - Network with project phase as vertex
  - Edges?
    - Tasks!




v1
v2
v3
v4
v5
v6
v7
v8
v11
a1
a2
a3
a4
a5
a5
a6
a7
a8
a9
a10
a11

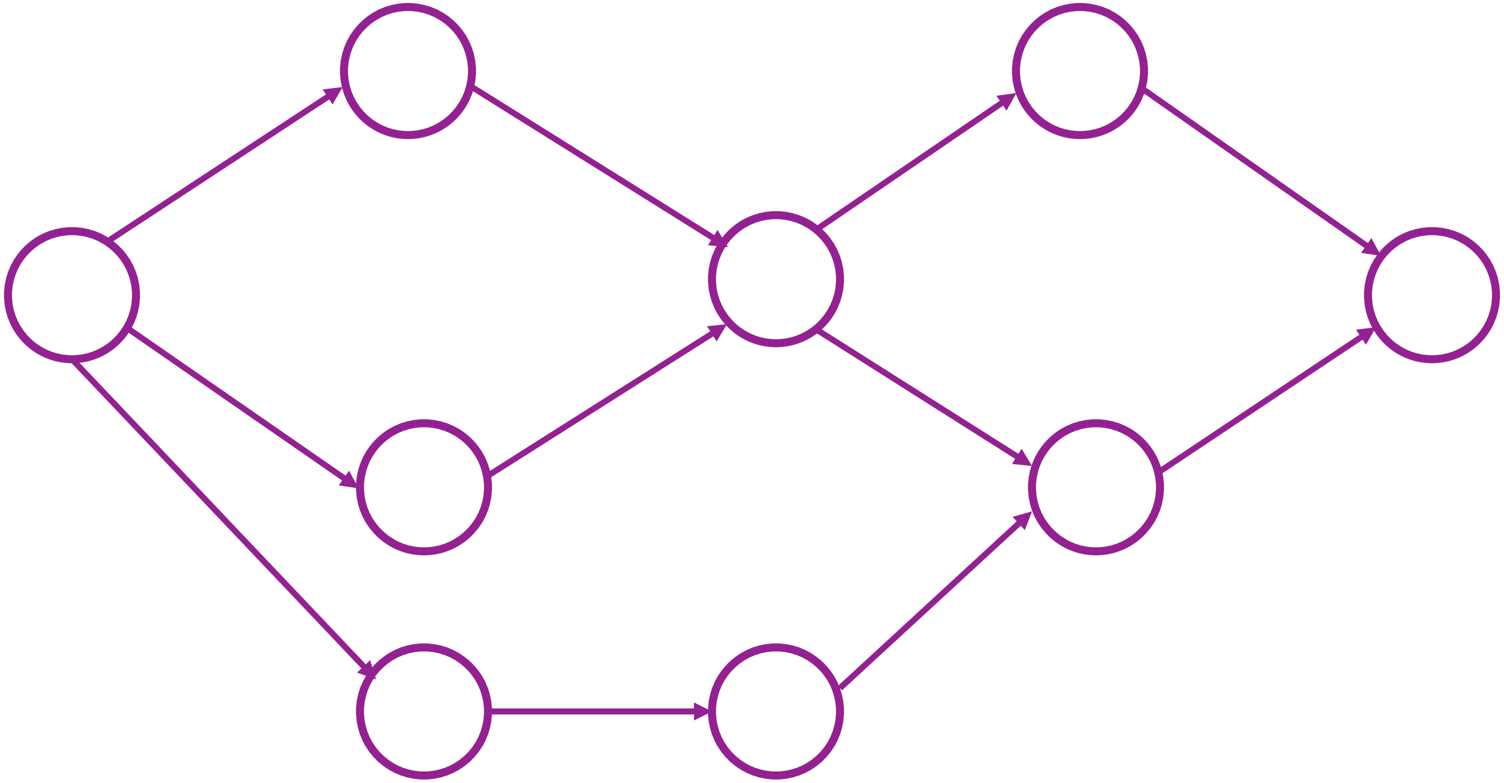
Activity-on-Edge (AOE) Networks

- directed edges --- tasks to be performed
- vertices --- events, signaling the completion of certain activities.
- activities represented by edges leaving a vertex cannot be started until the event at that vertex has occurred.
- an event occurs only when all activities entering it have been completed.

Revisit of Project planning

- Problem
  - How long at least to finish the project (all tasks)?
  - What tasks are critical to the finish time?
- Since activities in an AOE network can be carried out in parallel, the minimum time to complete the project is the length of the **longest path** from the start to the finish.
- A path of longest length is a **critical** path.

Another example



Critical Activity

- Critical activity
  - Edges in a critical path
  - Cannot delay
  - Starts as soon as possible
- How to identify critical tasks?
  - Given a project time
  - An earliest start time
  - A latest start time
  - If e(i) == l(i), then it is critical

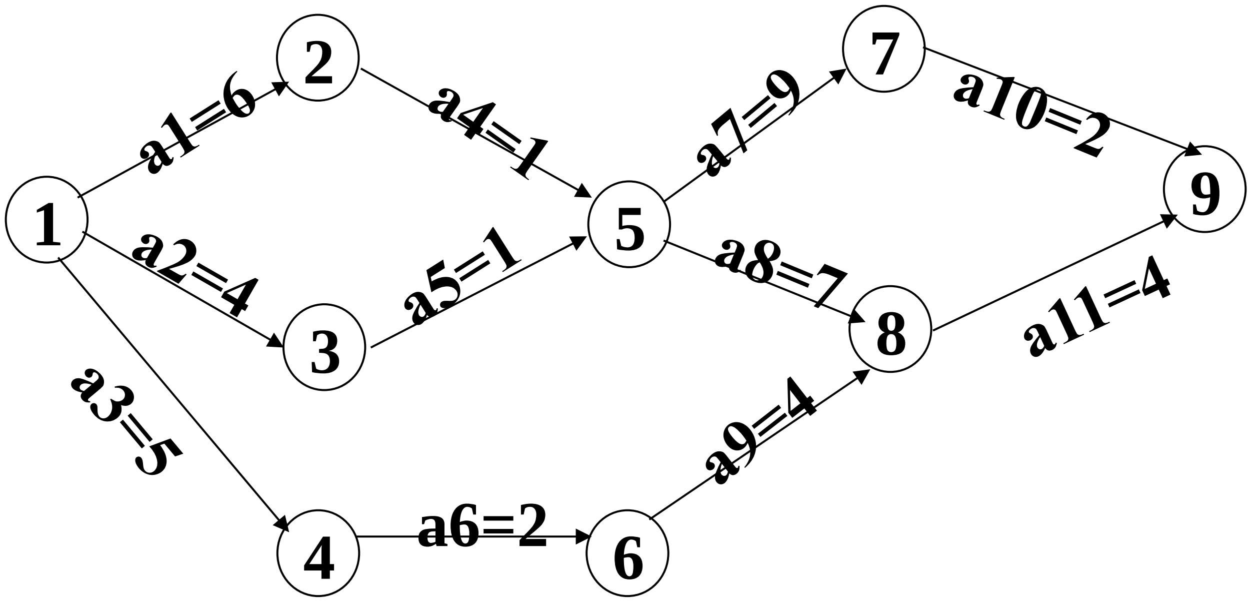
– Event 7 's latest time

Calculation of Early Activity Times

- **If a_i is edge <k, l>, then**

- **（1）e(i)=**
- **Ve(k)**
- **（2）l(i)=**
- **Vl(l)-dut(<k,l>)**

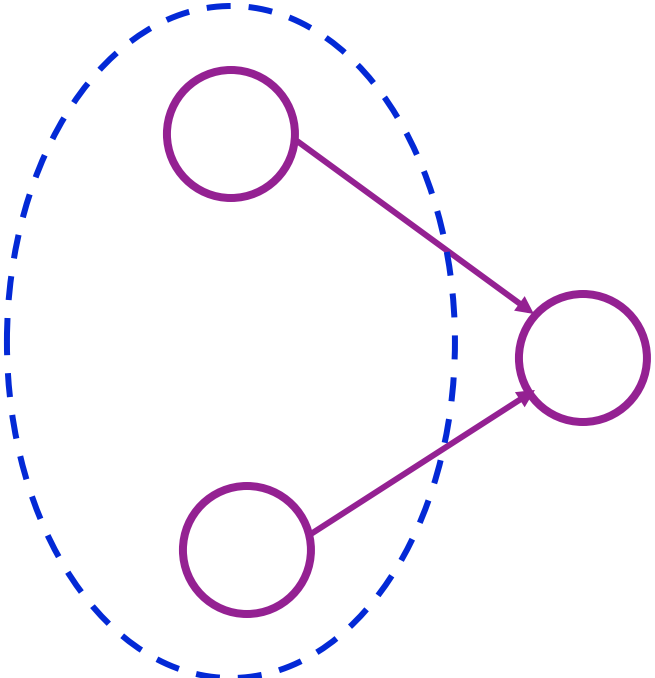
Calculation of Event Times



- E(1) = ?
  - 0
- E(2) = ?
  - 6
- E(3) = ?

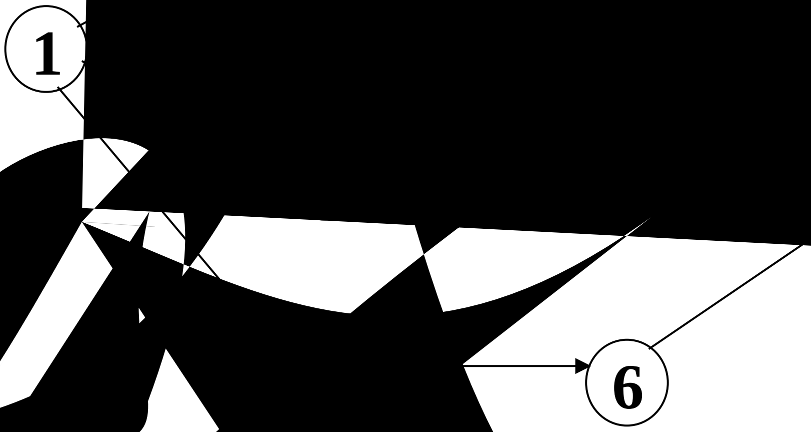
Calculation of Event Times

- P(j) is the set of all vertices adjacent to j.
- ee[0]=0 (suppose 0 is the start)
- ee[j]= max {ee[i]+duration of <i, j>}, i ∈P(j)



- Topological Order!

Cal



L(?

) = ?

L(9) – a10

Calculation of Event Times

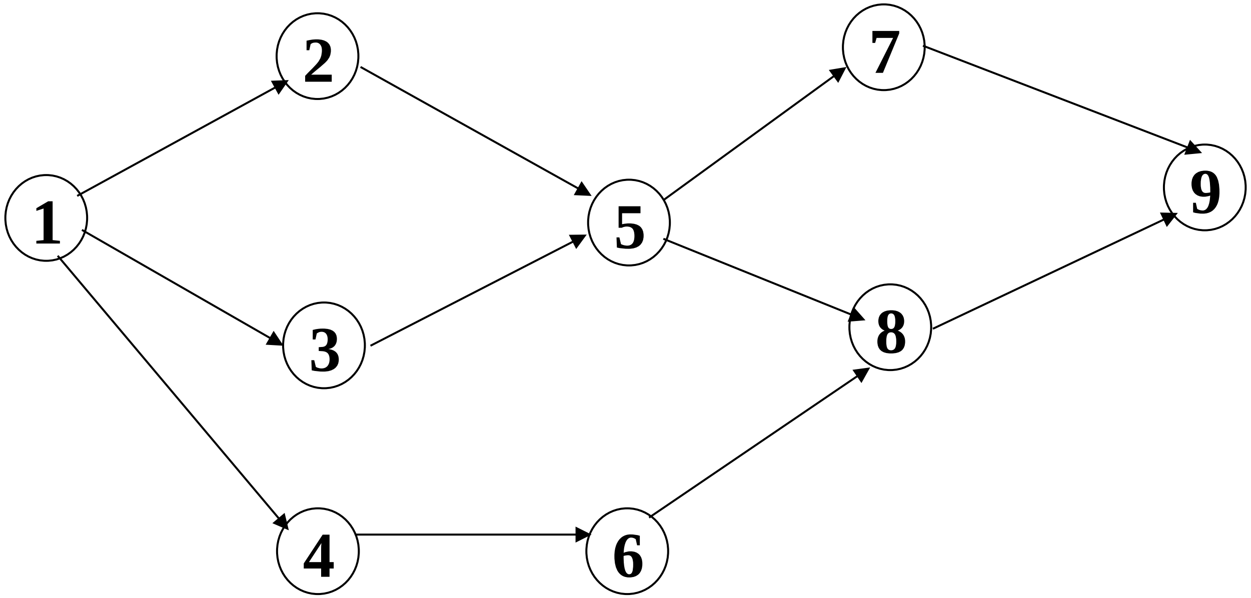
-

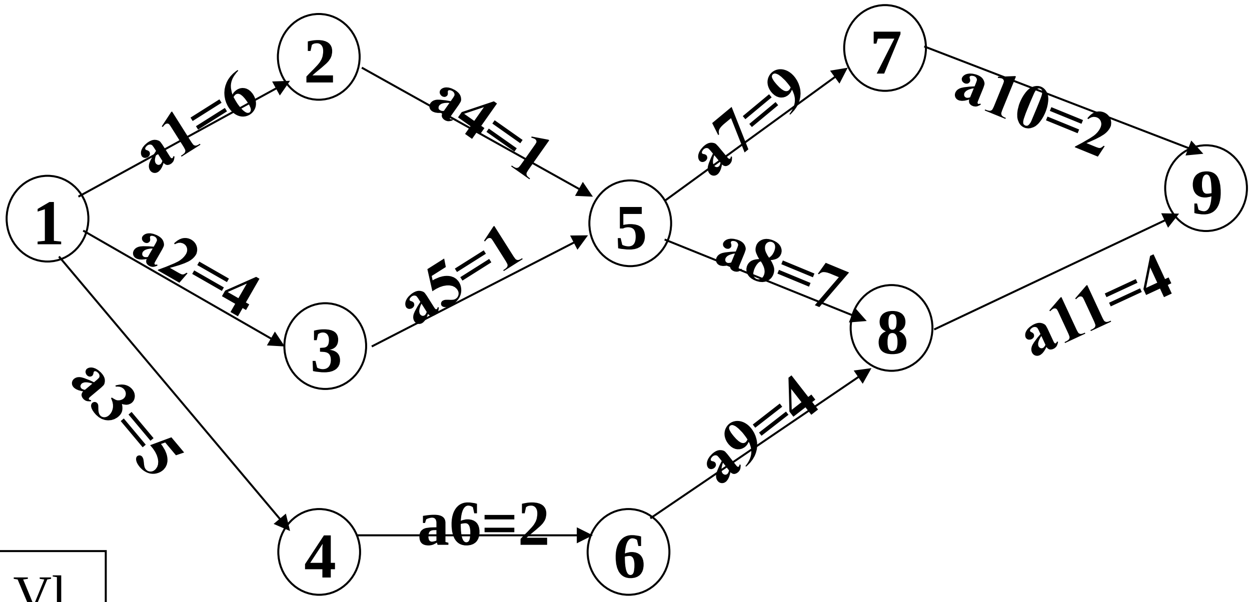
Revisit of Project planning

- Problem
  - How long at least to finish the project (all tasks)?
  - What tasks are critical to the finish time?
- **critical Path**
  - **Path length**
  - **Edges in path**

Critical Path

- Ve(i)
- Vl(i)
- E(i)
- L(i)
- L(i) – E(i)





Vertex	Ve	Vl
V1	0	0
V2	6	6
V3	4	6
V4	5	8
V5	7	7
V6	7	10
V7	16	16
V8	14	14
V9	18	18

Activity	e	l	l-e
a1	0	0	0 ✓
a2	0	2	2
a3	0	3	3
a4	6	6	0 ✓
a5	4	6	2
a6	5	8	3
a7	7	7	0 ✓
a8	7	7	0 ✓
a9	7	10	3
a10	16	16	0 ✓
a11	14	14	0 ✓

- struct Pair
- {
-     int vertex;
-     int dur;   //activity duration
- };

```
•  class LinkedGraph {
•  private:
•      Chain<Pair> *adjLists;
•      int *count, *t, *ee, *le;
•      int n;
•  public:
•      LinkedGraph (const int vertices) : {
•         if (vertices < 1) throw “Number of vertices must be > 0”;
•         n = vertices;
•         adjLists = new Chain<Pair>[n];
•         count = new int[n]; t = new int[n];
•         ee = new int[n]; le = new int[n];
•      };
•     void TopologicalOrder();
•     void EarliestEventTime();
•     void LatestEventTime();
•     void CriticalActivities();
•  };
```

```
void LinkedGraph::EarliestEventTime()
{ // assume a topological order has already been in t,
  // compute ee[j] according to t
   fill(ee, ee+n, 0);   // initialize ee
   for (i=0; i<n; i++) {
       int j=t[i];
      Chain<Pair>::ChainIterator ji=adjLists[j].begin();
      while (ji!=adjLists[j].end()) {
          int k=ji→vertex; //k is successor of j
          if (ee[k]<ee[j]+ji→dur) ee[k]=ee[j]+ji→dur;
            ji++;
      }
}
```

- **void** LinkedGraph::LatestEventTime()
- **{** // assume a topological order in t, ee has
-   // been computed, compute le[j] in the reverse order of t
-     fill(le, le+n, ee[n-1])**;** // initialize le
-     **for** (i=n-2**;** i>=0**;** i--) **{**
-

```
void LinkedGraph::CriticalActivities()
{ // compute e[i] and l[i], output critical activities
    int i=1; // the numbering counter for activities
    int u, v, e, l; // e, l are the earliest, latest start time of <u, v>
    for (u=0; u<n; u++) {  // scan the adjacency lists.
       Chain<Pair>::ChainIterator ui=adjLists[u].begin();
       while (ui!=adjLists[u].end()) {
          int v=ui→vertex; // <u, v> is an edge numbered i
          e=ee[u]; l=le[v]-ui→dur;
          if (l==e) cout <<“a”<<i<<“<“<<u<<“,”<<v<<“>”
                                 <<“is a critical activity”<<endl;
            ui++; i++;
       }
    }
}
```

Graph

- Definitions
- Representations
- Search algorithms
- Spanning tree
- Shortest path
- AOV
- AOE